MATH6103 Differential & Integral Calculus
MATH6500 Elementary Mathematics for Engineers

Problem Sheet 9 Solutions

1) Solve the following second order ODEs.

* b) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = e^x$

$$\begin{aligned}
\text{Auxillary equation is } \lambda^2 + \lambda - 6 &= 0 \\
(\lambda - 2)(\lambda + 3) &= 0 \\
\lambda &= -3, 2 \\
\text{The solution of } \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y &= 0 \text{ is} \\
y &= Ae^{2x} + Be^{-3x}
\end{aligned}$$

$$\begin{aligned}
\text{Try } y &= Ce^x \\
\frac{dy}{dx} &= Ce^x \\
\frac{d^2y}{dx^2} &= Ce^x \\
Ce^x + Ce^x - 6Ce^x &= e^x \\
-4C &= 1 \\
C &= -\tfrac{1}{4}
\end{aligned}$$

$$y = Ae^{2x} + Be^{-3x} - \tfrac{1}{4}e^x$$

* c) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = e^{2x}$

$$\begin{aligned}
\text{From (1b), the solution of } \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y &= 0 \text{ is} \\
y &= Ae^{2x} + Be^{-3x}
\end{aligned}$$

$$\begin{aligned}
\text{Try } y &= Cxe^{2x} \\
\frac{dy}{dx} &= Ce^{2x} + 2Cxe^{2x} \\
\frac{d^2y}{dx^2} &= 4Ce^{2x} + 4Cxe^{2x} \\
4Ce^{2x} + 4Cxe^{2x} + Ce^{2x} + 2Cxe^{2x} - 6Cxe^{2x} &= e^{2x} \\
5Ce^{2x} &= e^{2x} \\
C &= \tfrac{1}{5}
\end{aligned}$$

$$y = Ae^{2x} + Be^{-3x} + \tfrac{1}{5}xe^{2x}$$

* f) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 18y = 34e^x$

Auxillary equation is $\lambda^2 + 6\lambda + 18 = 0$

$$\begin{aligned} \lambda &= \frac{-6 \pm \sqrt{6^2 - 4 \times 18}}{2} \\ &= \frac{-6 \pm \sqrt{-36}}{2} \\ &= \frac{-6 \pm 6\text{i}}{2} \\ &= -3 \pm 3\text{i} \end{aligned}$$

The solution of $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$ is

$$y = e^{-3x}(A\sin(3x) + B\cos(3x))$$

$$\begin{aligned} \text{Try } y &= Ce^x \\ \frac{dy}{dx} &= Ce^x \\ \frac{d^2y}{dx^2} &= Ce^x \\ Ce^x + 6Ce^x + 18Ce^x &= 34e^x \\ 25C &= 34 \\ C &= \tfrac{34}{25} \end{aligned}$$

$$y = e^{-3x}(A\sin(3x) + B\cos(3x)) + \tfrac{34}{25}e^x$$

2) Solve the following second order ODEs with boundary conditions.

* a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 6e^{3x}, \quad y(0) = 3 \quad y'(0) = 8$

$$\begin{aligned} \lambda^2 - 3\lambda + 2 &= 0 \\ \lambda &= 2, 1 \\ y &= Ae^x + Be^{2x} \end{aligned}$$

$$\begin{aligned} \text{Try } y &= Ce^{3x} \\ \frac{dy}{dx} &= 3Ce^x \\ \frac{d^2y}{dx^2} &= 9Ce^x \\ 9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} &= 6e^{3x} \\ C &= 3 \end{aligned}$$

$$y = Ae^x + Be^{2x} + 3e^{3x}$$

$$y(0) = 3 \implies \begin{aligned} 3 &= Ae^0 + Be^0 + 3e^0 \\ 3 &= A + B + 3 \\ A &= -B \end{aligned}$$

$$y'(0) = 8 \implies \begin{aligned} 8 &= Ae^0 + 2Be^0 + 9e^0 \\ 8 &= A - 2A + 9 \\ A &= 1 \\ B &= -1 \end{aligned}$$

$$y = e^x - e^{2x} + 3e^{3x}$$

* b) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x, \quad y(0) = 3, \quad y'(0) = 3 \quad y(1) = \dfrac{7e}{2}$

$$\begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ \lambda &= 1, 1 \\ y &= Ae^x + Bxe^x \end{aligned}$$

$$\begin{aligned} \text{Try } y &= Cx^2e^x \\ \frac{dy}{dx} &= Cx^2e^x + 2Cxe^x \\ \frac{d^2y}{dx^2} &= Cx^2e^x + 4Cxe^x + 2Ce^x \\ Cx^2e^x + 4Cxe^x + 2Ce^x - 2Cx^2e^x - 4Cxe^x + Cx^2e^x &= e^x \\ 2Ce^x &= e^x \\ C &= \tfrac{1}{2} \end{aligned}$$

$$y = Ae^x + Bxe^x + \tfrac{1}{2}x^2e^x$$

$$y(0) = 3 \implies \begin{aligned} 3 &= Ae^0 + B \times 0 \times e^0 + \tfrac{1}{2} \times 0^2e^0 \\ 3 &= A \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= (A + B)e^x + (B + 1)xe^x + \tfrac{1}{2}x^2e^x \\ &= (3 + B)e^x + (B + 1)xe^x + \tfrac{1}{2}x^2e^x \end{aligned}$$

$$y'(0) = 3 \implies \begin{aligned} 3 &= (3 + B)e^0 + (B + 1) \times 0 \times e^0 + \tfrac{1}{2} \times 0^2 \times e^0 \\ 0 &= B \end{aligned}$$

$$\text{Or } y(1) = \frac{7e}{2} \implies \begin{aligned} \frac{7e}{2} &= 3e^1 + B \times 1 \times e^1 + \tfrac{1}{2} \times 1^2e^1 \\ \frac{7e}{2} &= 3e + Be + \tfrac{1}{2}e \\ 0 &= B \end{aligned}$$

$$y = 3e^x + \tfrac{1}{2}x^2e^x$$

* 3a) Use Euler's Method with three steps to approximate $y(1)$ when $y(0) = 3$ and

$$\frac{dy}{dx} = y + 6x.$$

Give your answer as a fraction.

$h = \frac{1}{3}$

x	y	$\frac{dy}{dx}$ $(= y + 6x)$	$y + h\frac{dy}{dx}$
0	3	3	4
$\frac{1}{3}$	4	6	6
$\frac{2}{3}$	6	10	$9\frac{1}{3}$
1	$9\frac{1}{3}$	-	-

* 3b) Give one way in which the accuracy of the answer to (3a) could be improved.

Make h smaller / use more steps.