4.2.2 Integrating factors

Homogenous first order linear ODEs can be solved by seperating the variables. Non-homogenous ODEs cannot. In this section we will look at how to solve non-homogenous first order linear ODEs.

A first order linear ODE looks like

$$\frac{dy}{dx} + g(x)y = f(x).$$

When $f(x)$ is not zero, the ODE is non-homogenous. First order non-homogenous ODEs can be solved using integrating factors.

Definition
The **integrating factor** of the ODE

$$\frac{dy}{dx} + g(x)y = f(x).$$

is

$$\exp\left(\int g(x)\ dx\right).$$

When finding the integrating factor we can ignore the constant of integration.

If we multiply each term by the integrating factor, it should make our equation easier to solve. To solve

$$\frac{dy}{dx} + g(x)y = f(x),$$

let

$$T(x) = \int g(x)\ dx,$$

so that the integrating factor is

$$e^{T(x)}.$$

Multiplying through by the integrating factor gives

$$e^{T(x)}\frac{dy}{dx} + e^{T(x)}g(x)y = e^{T(x)}f(x),$$

By the product rule, we can find the derivative of $e^{T(x)}y$:

$$\begin{aligned}\frac{d}{dx}\left(e^{T(x)}y\right) &= e^{T(x)}\frac{dy}{dx} + y\frac{d}{dx}\left(e^{T(x)}\right)\\ &= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}\frac{d}{dx}\left(T(x)\right)\\ &= e^{T(x)}\frac{dy}{dx} + ye^{T(x)}g(x)\end{aligned}$$

This is exactly what we had on the left hand side of the ODE after multiplying by the integrating factor. Therefore:

$$\begin{aligned}\frac{d}{dx}\left(e^{T(x)}y\right) &= e^{T(x)}f(x)\\ e^{T(x)}y &= \int e^{T(x)}f(x)\ dx\\ y &= \frac{\int e^{T(x)}f(x)\ dx}{e^{T(x)}}\end{aligned}$$

If we know how to integrate $e^{T(x)}f(x)$, then we can use this method to solve the ODE.

Example
Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x.$$

[note that $f(x, y) = x - (y/x)$ can't be separated.]
The integrating factor is:

$$\begin{aligned}\exp\left(\int \frac{1}{x}\ dx\right) &= \exp\left(\ln x\right)\\ &= x\end{aligned}$$

Multiplying through by the integrating factor gives:

$$x\frac{dy}{dx} + y = x^2$$

Notice that:

$$\frac{d}{dx}\left(xy\right) = x\frac{dy}{dx} + y$$

Therefore:

$$\begin{aligned}\frac{d}{dx}\left(xy\right) &= x^2\\ xy &= \int x^2\ dx\\ &= \frac{x^3}{3} + c\\ y &= \frac{x^2}{3} + \frac{c}{x}\end{aligned}$$

Example

$$
\begin{aligned}
\frac{dy}{dx} + xy &= x \\
\exp\left(\int x\ dx\right)\frac{dy}{dx} + \exp\left(\int x\ dx\right)xy &= \exp\left(\int x\ dx\right)x \\
\exp\left(\frac{1}{2}x^2\right)\frac{dy}{dx} + \exp\left(\frac{1}{2}x^2\right)xy &= \exp\left(\frac{1}{2}x^2\right)x \\
\frac{d}{dx}\left(\exp\left(\frac{1}{2}x^2\right)y\right) &= \exp\left(\frac{1}{2}x^2\right)x \\
\exp\left(\frac{1}{2}x^2\right)y &= \int \exp\left(\frac{1}{2}x^2\right)x\ dx \\
\exp\left(\frac{1}{2}x^2\right)y &= \exp\left(\frac{1}{2}x^2\right) \\
y &= \frac{\exp\left(\frac{1}{2}x^2\right) + c}{\exp\left(\frac{1}{2}x^2\right)} \\
y &= 1 + c\exp\left(-\frac{1}{2}x^2\right)
\end{aligned}
$$

Or

$$y = 1 + ce^{-\frac{1}{2}x^2}$$

Example
Solve the initial-value problem

$$y' = y + x^2, \quad y(0) = 1.$$

First, find the general solution:

$$
\begin{aligned}
\frac{dy}{dx} - y &= x^2 \\
\exp\left(\int -1\ dx\right)\frac{dy}{dx} - \exp\left(\int -1\ dx\right)y &= \exp\left(\int -1\ dx\right)x^2 \\
e^{-x}\frac{dy}{dx} - e^{-x}y &= e^{-x}x^2 \\
\frac{d}{dx}\left(e^{-x}y\right) &= e^{-x}x^2 \\
e^{-x}y &= \int e^{-x}x^2\ dx \\
&= -e^{-x}x^2 + \int 2xe^{-x}\ dx \\
&= -e^{-x}x^2 - 2xe^{-x} + \int 2e^{-x}\ dx \\
&= -e^{-x}x^2 - 2xe^{-x} - 2e^{-x} + c \\
y &= -x^2 - 2x - 2 + ce^{x}
\end{aligned}
$$

Now we must use the initial condition to find c:

$$\begin{aligned} y &= -x^2 - 2x - 2 + ce^x \\ 1 &= -0^2 - 2 \cdot 0 - 2 + ce^0 \\ 1 &= -2 + c \\ 3 &= c \end{aligned}$$

Therefore the solution to the problem is

$$y = -x^2 - 2x - 2 + 3e^x.$$