3.3.4 Trigonometric substitution

Example
We know that

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c,$$

since

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}.$$

Actually, we can work out this integral by a substitution like $x = \sin u$ because we know that

$$1 - \sin^2 u = \cos^2 u,$$

and

$$\frac{dx}{du} = \cos u \quad \text{or} \quad dx = \cos u \; du.$$

Thus, we calculate the integral as

$$\int \frac{1}{\sqrt{1-x^2}} \; dx = \int \frac{\cos u}{\sqrt{1-\sin^2 u}} \; du = \int \; du = u + c = \sin^{-1} x + c.$$

Example

$$\int \frac{1}{1+x^2} \; dx = \tan^{-1} x + c, \quad \text{since} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

Let us try the following

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta} = 1 + \tan^2\theta$$

$$dx = (1+\tan^2\theta) \; d\theta$$

$$\int \frac{1}{1+x^2} \; dx = \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta) \; d\theta = \int \; d\theta = \theta + c = \tan^{-1} x + c.$$

Example

$$\int \frac{1}{1+2x^2} \; dx$$

This is similar to the previous example. If we try

$$\sqrt{2}x = \tan\theta$$

$$dx = \frac{1}{\sqrt{2}}(1+\tan^2\theta) \; d\theta$$

$$\theta = \tan^{-1}(\sqrt{2}x)$$

$$\begin{aligned}
\int \frac{1}{1+2x^2}\ dx &= \int \frac{1}{1+(\sqrt{2}x)^2}\ dx \\
&= \frac{1}{\sqrt{2}} \int \frac{1}{1+\tan^2\theta}(1+\tan^2\theta)\ d\theta \\
&= \frac{1}{\sqrt{2}}\theta + c \\
&= \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}x) + c.
\end{aligned}$$