2.9 Real life examples

Example
You are required to design an open box with a square base and a total volume of 4m^3 using the least amount of materials.
We are hunting for particular dimensions of the box. Lets give them names, so we will call the length of the base edges a and let the height of the box be called b.

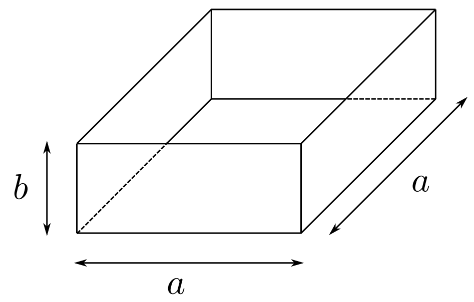

Figure 2.15: Square based open top box.

In terms of a and b, the volume of the box is

$$a^2 b = 4 \ (\text{m}^3).$$

We want to minimise the amount of material, so we must look at how much material is used. We have a base of area a^2 and 4 sides, each with area ab. So the total area is

$$a^2 + 4ab.$$

We want to find the minimum value of $a^2 + 4ab$, subject to the condition $a^2 b = 4$ and more obviously $a > 0$, $b > 0$. From the condition, we see that $b = 4/a^2$, so we can rewrite the amount of material in terms of a:

$$a^2 + 4ab = a^2 + 4a\left(\frac{4}{a^2}\right) = a^2 + \frac{16}{a}.$$

So now we can write a function for the material area solely in terms of one of the lengths of the box (the other is now fixed by using the condition). That is

$$f(a) = a^2 + \frac{16}{a}.$$

Now written like a function, it is easy to see how we would minimise the material, that is by minimising the function $f(a)$. So we have to calculate $f'(a)$, which is

$$f'(a) = 2a - \frac{16}{a^2},$$

and now we simply need to see when $f'(a) = 0$:

$$2a - \frac{16}{a^2} = 0 \quad \Longrightarrow \quad 2a^3 = 16 \quad \Longrightarrow \quad a = 2.$$

The only real number at which $f'(a) = 0$ is $a = 2$. The function $f'(a)$ makes sense except at $a = 0$, which is outside the range (since $a > 0$). Differentiating $f'(a)$ gives

$$f''(a) = 2 + \frac{32}{a^3}.$$

This is positive for the whole domain (and at $a = 2$), and so this point is a minimum.

Example

We have a pair of islands, island 1 and island 2, 20km and 10km away from a straight shore, respectively. The perpendiculars from the islands to the shore are 30km apart (along the shore). What is the quickest way between the two islands that goes via the shore?

We are trying to find a point along the shore, which we want to visit when going from one island to the other. This point can specified by the distance from the perpendicular of island 1, call this distance x.

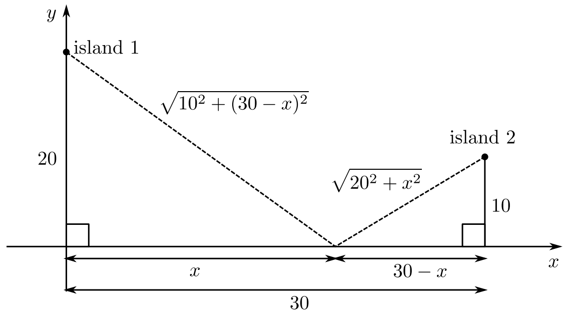

Figure 2.16: The set up of islands relative to the shore (x-axis).

The total distance $D(x)$ to be travelled is given by the formula

$$D(x) = \sqrt{20^2 + x^2} + \sqrt{10^2 + (30 - x)^2}.$$

On geometric grounds, we see that $0 \leq x \leq 30$, and $D(x)$ is differentiable on this domain.

$$\begin{aligned} D'(x) &= \frac{1}{2}(20^2 + x^2)^{-\frac{1}{2}} \cdot 2x + \frac{1}{2}\left(10^2 + (30 - x)^2\right)^{-\frac{1}{2}} 2(30 - x) \cdot -1 \\ &= \frac{x}{\sqrt{20^2 + x^2}} - \frac{30 - x}{\sqrt{(30 - x)^2 + 10^2}}. \end{aligned}$$

To find the minimum distance we set $D'(x) = 0$, so we have

$$\frac{x}{\sqrt{20^2 + x^2}} = \frac{30 - x}{\sqrt{(30 - x)^2 + 10^2}}.$$

Squaring both sides we get

$$\frac{x^2}{20^2 + x^2} = \frac{(30 - x)^2}{(30 - x)^2 + 10^2} \quad \Longrightarrow \quad (x^2)[(30 - x)^2 + 10^2] = (30 - x)^2(20^2 + x^2)$$

Expanding the brackets we see

$$10^2x^2 = 20^2(30 - x)^2.$$

Take the square root of both sides of the last equation gives

$$10x = \pm 20(30 - x).$$

Since $30 - x \geq 0$ and $x \geq 0$, the negative sign is not possible, so

$$10x = 20(30 - x) \quad \Longrightarrow \quad x = 20.$$

The only possible places where minimum can occur are

$$x = 0, \quad x = 20, \quad x = 30,$$

where

$$D(0) = 20 + \sqrt{10^2 + 30^2} \approx 51.6,$$
$$D(20) = 20\sqrt{2} + 10\sqrt{2} = 30\sqrt{2} \approx 42.4,$$
$$D(30) = \sqrt{20^2 + 30^2} + 10 \approx 46.06.$$

So the minimum value does occur at $x = 20$.

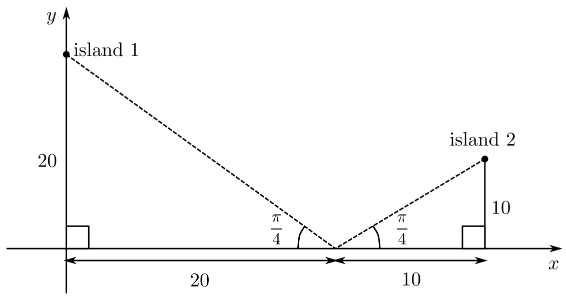

Figure 2.17: Optimum route from island 1 to island 2, arriving and leaving shore at an angle of $\pi/4$ radians.

The optimal route is to leave the shore at the same angle of arrival.
Could we have deduced that this route was the shortest without calculus? Yes! We could have reflected island 2 in the shore line to obtain an imaginary island. Then it is easy to see that the shortest route from island 1 to the imaginary island is a straight line.

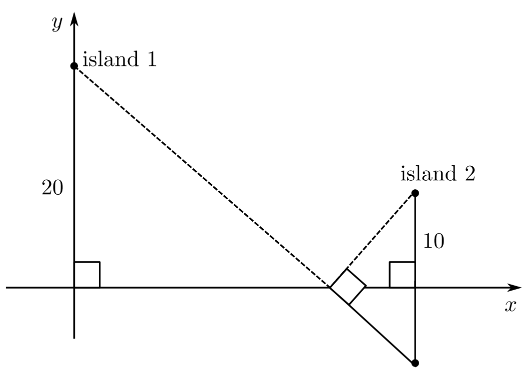

Figure 2.18: Sketch of alternative method of calculating shortest distance between the islands, using simple geometric properties, gaining same result.

2.9.1 Exponential growth and decay

Let $y = f(t)$ represent some physical quantity, such as the volume of a substance, the population of a certain species or the mass of a decaying radioactive substance. We want to measure the growth or decay of $f(t)$.

In many applications, the rate of growth (or decay) of a quantity is proportional to the quantity. In other "words":

$$\frac{dy}{dt} = \alpha y, \quad \alpha = \text{constant}.$$

This is a **differential equation** whose solution is

$$y(t) = ce^{\alpha t},$$

where constant c is determined by an **initial condition**, say, $y(0) = y_0$ (given). Therefore we have

$$y(t) = y_0 e^{\alpha t}.$$

This means that if you start with y_0, after time t you have $y(t)$.

If $\alpha > 0$, the quantity is increasing (growth). If $\alpha < 0$, the quantity is decreasing (decay).

We will study differential equations in more detail later in the course.

Radioactive decay

Atoms of elements which have the same number of protons but differing numbers of neutrons are referred to as isotopes of each other. Radioisotopes are isotopes that decompose and in doing so emit harmful particles and/or radiation.

It has been found experimentally that the atomic nuclei of so-called radioactive elements spontaneously decay. They do it at a characteristic rate.

If we start with an amount M_0 of an element with decay rate λ (where $\lambda > 0$), then after time t, the amount remaining is

$$M = M_0 e^{-\lambda t}.$$

This is the radioactive decay equation. The proportion left after time t is

$$\frac{M}{M_0} = e^{-\lambda t},$$

and the proportion decayed is

$$1 - \frac{M}{M_0} = 1 - e^{-\lambda t},$$

Carbon dating

Carbon dating is a technique used by archeologists and others who want to estimate the age of certain artefacts and fossils they uncover. The technique is based on certain properties of the carbon atom.

In its natural state, the nucleus of the carbon atom C^{12} has 6 protons and 6 neutrons. The isotope carbon-14, C^{14}, has 6 protons and 8 neutrons and is radioactive. It decays by beta emission.

Living plants and animals do not distinguish between C^{12} and C^{14}, so at the time of death, the ratio C^{12} to C^{14} in an organism is the same as the ratio in the atmosphere. However, this ratio changes after death, since C^{14} is converted into C^{12} but no further C^{14} is taken in.

Example
Half-lives: how long before half of what you start with has decayed? When do we get $M = \frac{1}{2}M_0$? We need to solve

$$\frac{M}{M_0} = \frac{1}{2} = e^{-\lambda t},$$

taking logarithms of both sides gives

$$\ln\left(\frac{1}{2}\right) = -\lambda t \qquad \Longrightarrow \qquad t = \frac{\ln(2)}{\lambda}.$$

So, the half-life, $T_{\frac{1}{2}}$ is given by

$$T_{\frac{1}{2}} = \frac{\ln(2)}{\lambda}.$$

If λ is in "per year", then $T_{\frac{1}{2}}$ is in years.

Example
Carbon-14 (C^{14}) exists in plants and animals, and is used to estimate the age of certain fossils uncovered. It is also used to trade metabolic pathways. C^{14} is radioactive and has a decay rate of $\lambda = 0.000125$ (per year). So we can calculate its half-life as

$$T_{\frac{1}{2}} = \frac{\ln 2}{0.000125} \approx 5545 \text{ years.}$$

Example
A certain element has $T_{\frac{1}{2}}$ of 10^6 years

1. What is the decay rate?

$$\lambda = \frac{\ln 2}{T_{\frac{1}{2}}} \approx \frac{0.693}{10^3} \approx 7 \times 10^{-7} \text{ (per year).}$$

2. How much of this will have decayed after 1000 years? The proportion remaining is
$$\frac{M}{M_0} = e^{-\lambda t} = e^{-7\times 10^{-7}\times 10^3} = e^{-7\times 10^{-4}} \approx 0.9993.$$
The proportion decayed is
$$1 - \frac{M}{M_0} \approx 1 - 0.9993 = 0.0007.$$

3. How long before 95% has decayed?
$$\frac{M}{M_0} = 1 - 0.95 = 0.05 = e^{-7\times 10^{-7} t},$$
taking logarithms of both sides we have
$$\ln(0.05) = -7 \times 10^{-7} t$$
which implies
$$t = \frac{\ln(0.05)}{-7\times 10^{-7}} \approx \frac{-2.996}{-7\times 10^{-7}} \approx 4.3 \times 10^6 \text{ (years)}.$$

WARNING: Half-life $T_{\frac{1}{2}}$ of a particular element does not mean that in $2 \times T_{\frac{1}{2}}$, the element will completely decay.

Population growth

Example
Suppose a certain bacterium divides each hour. Each hour the population doubles:

Hours	1	2	3	4	5	...
Population	2	4	8	16	32	...

After t hours you have 2^t times more bacteria than what you started with. In general, we write it as an exponential form.

If a population, initially P_0 grows exponentially with growth rate λ (where $\lambda > 0$), then at time t, the population is
$$P(t) = P_0 e^{\lambda t}.$$

Example
Bacterium divides every hour.

1. What is the growth rate?

   We know that when $t = 1$ hour, we are supposed to have
$$P = 2P_0,$$

so

$$2P_0 = P_0 e^{\lambda \cdot 1} \quad \Longrightarrow \quad 2 = e^{\lambda} \quad \Longrightarrow \quad \lambda = \ln 2 \approx 0.693.$$

2. How long for 1 bacterium to become 1 billion?

$$P_0 = 1, \quad \lambda = 0.693, \quad P = 10^9,$$

therefore we may write

$$P = P_0 e^{\lambda t} \quad \Longrightarrow \quad 10^9 = e^{0.693t},$$

taking logarithms of both sides and re-arranging for t, we have

$$t = \frac{9 \ln 10}{0.693} \approx 30 \text{ hours.}$$

Interest rate

An annual interest rate of 5% tells you that £100 investment at the start of the year grows to £105. Each subsequent year you leave your investment, it will be multiplied by the factor 1.05.

In general, if you initially invest M_0 (amount) with an annual interest rate r (given as percentage/100), then after t years you have

$$A = M_0(1+r)^t,$$

where A is the future value. We could write this as an exponential as follows:

$$A(t) = M_0 e^{\lambda t} = M_0(1+r)^t.$$

Taking logarithms we have

$$\lambda t = t \ln(1+r),$$

so we may write

$$A(t) = M_0 e^{\ln(1+r)t}.$$