2.8 Polar co-ordinates

A circle with radius r can be written as $x^2 + y^2 = r^2$. However, it is more natural to think of a circle as $r =$ constant.

We can write a circle in this more natural way by using polar co-ordinates.

Definition
In **polar co-ordinates**, we give the distance from the origin, r, and the angle made with the x-axis, *theta*, as shown on the diagram below.

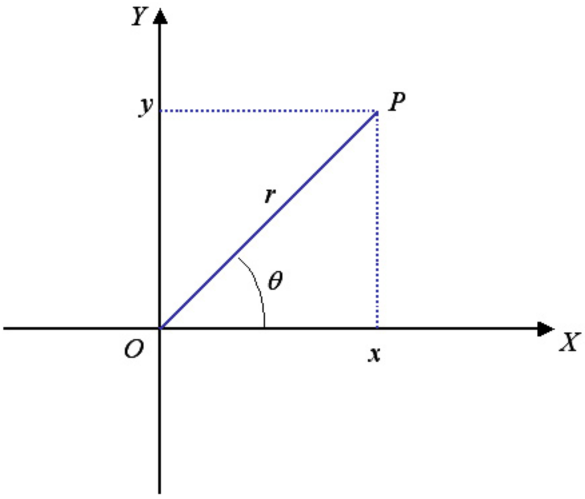

Figure 2.12: Polar co-ordinates are given by r and θ.

Shapes can be defined by writing r in terms of θ.

The Cartesian (normal) co-ordinates can be written in terms of the polar co-ordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Example
The following cardioid can be most easily written in polar co-ordinates as $r = 1 - \cos\theta$.

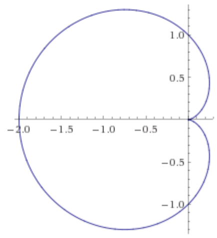

Figure 2.13: The cardioid $r = 1 - \cos\theta$.

If written in Cartesian co-ordinates, it would be

$$x^2 + y^2 + x = \sqrt{x^2 + y^2}.$$

2.8.1 Implicit Differentiation

The chain rule can be rearranged to give:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

This can be used whenever y and x are known in terms of a parameter t.

Example
If $x = t^2 + 4$ and $y = e^t$ then:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{e^t}{2t}$$

In most cases, the answer can be left in terms of t.

Example
To find the gradient of the cardioid given by the polar equation $r = 1 - \cos\theta$, we must first write x and y in terms of θ.

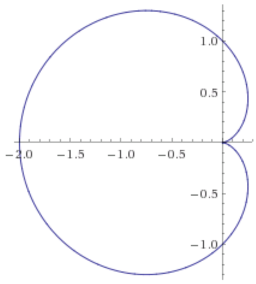

Figure 2.14: The cardioid with polar equation $r = 1 - \cos\theta$.

$$\begin{aligned} x &= r\cos\theta \\ &= (1-\cos\theta)\cos\theta \\ &= \cos\theta - \cos^2\theta \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= (1-\cos\theta)\sin\theta \\ &= \sin\theta - \sin\theta\cos\theta \end{aligned}$$

Now we can differentiate:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\cos\theta + \sin^2\theta - \cos^2\theta}{-\sin\theta + 2\cos\theta\sin\theta} \end{aligned}$$