MATH6103 Differential & Integral Calculus

MATH6500 Elementary Mathematics for Engineers

Problem Sheet 7 Solutions

1) A car is accelerating down a straight road. Its velocity, vm/s, at time t seconds is given by:

$$v(t) = 10t$$

How far has the car travelled when it reaches 30m/s?

> $v(3) = 30$, so the car reaches 30m/s after 3 seconds. The distance is given by:
>
> $$\begin{aligned}\int_0^3 10t\ dt &= \left[5t^2\right]_0^3 \\ &= \left(5\cdot 3^2\right) - \left(5\cdot 0^2\right) \\ &= 5\cdot 9 - 0 \\ &= 45\end{aligned}$$

2) Find the area of the finite region between $y = e^x - 1$ and $y = (e^2 - 1)x$.

> The area under $y = e^x - 1$ is:
>
> $$\begin{aligned}\int_0^1 e^x - 1\ dx &= \left[e^x - x\right]_0^1 \\ &= \left(e^1 - 1\right) - \left(e^0 - 0\right) \\ &= e - 1 - 1 + 0 \\ &= e - 2\end{aligned}$$
>
> The area under $y = (e^2 - 1)x$ is:
>
> $$\begin{aligned}\int_0^1 (e^2 - 1)x\ dx &= \left[\frac{1}{2}(e^2 - 1)x^2\right]_0^1 \\ &= \left(\frac{1}{2}(e^2 - 1)\cdot 1^2\right) - \left(\frac{1}{2}(e^2 - 1)\cdot 0^2\right) \\ &= \frac{1}{2}(e^2 - 1)\end{aligned}$$
>
> The area between the curves is:
>
> $$\begin{aligned}&(e^2 - 1) - (e - 2) \\ =&e^2 - e + 1\end{aligned}$$

3) Use the trapezium method with 4 trapeziums to estimate $\displaystyle\int_0^2 x^2\ dx$.

$$
\begin{aligned}
h &= \frac{b-a}{4} \\
&= \frac{2-0}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{array}{ll}
x_0 = 0 & y_0 = 0 \\
x_1 = \frac{1}{2} & y_1 = \frac{1}{4} \\
x_2 = 1 & y_2 = 1 \\
x_3 = \frac{3}{2} & y_3 = \frac{9}{4} \\
x_4 = 2 & y_4 = 4
\end{array}
$$

Using the trapezium rule,

$$
\begin{aligned}
\int_0^2 x^2 \, dx \approx& \frac{h}{2}(y_0 + y_4) + h(y_1 + y_2 + y_3) \\
&= \frac{1}{4}(0+4) + \frac{1}{2}(\frac{1}{4} + 1 + \frac{9}{4}) \\
&= 1 + \frac{1}{2}(1 + \frac{10}{4}) \\
&= 1 + \frac{1}{2}(1 + \frac{5}{2}) \\
&= 1 + \frac{1}{2}(\frac{7}{2}) \\
&= 1 + \frac{7}{4} \\
&= \frac{11}{4} \text{ or } 2.75
\end{aligned}
$$

4a) Use the trapezium method with 4 trapeziums to estimate $\displaystyle\int_0^1 \frac{1}{1+x^2} \, dx$.

$$
\begin{aligned}
h &= \frac{b-a}{4} \\
&= \frac{1-0}{4} \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{array}{ll}
x_0 = 0 & y_0 = 1 \\
x_1 = \frac{1}{4} & y_1 = \frac{16}{17} \\
x_2 = \frac{1}{2} & y_2 = \frac{4}{5} \\
x_3 = \frac{3}{4} & y_3 = \frac{16}{25} \\
x_4 = 1 & y_4 = \frac{1}{2}
\end{array}
$$

Using the trapezium rule,

$$\begin{aligned}\int_0^2 x^2\ dx \approx& \frac{h}{2}(y_0+y_4)+h(y_1+y_2+y_3)\\ &= \frac{1}{8}(1+\frac{1}{2})+\frac{1}{4}(\frac{16}{17}+\frac{4}{5}+\frac{16}{25})\\ &= 0.7827\end{aligned}$$

4b) Find $\int_0^1 \frac{1}{1+x^2}\ dx$ exactly.

Let $x = \tan u$

$$\begin{aligned}\frac{dx}{du} &= \sec^2 u\\ dx &= \sec^2 u\ du\end{aligned}$$

Therefore:

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\ dx &= \int_{x=0}^{x=1} \frac{1}{1+\tan^2 u}\sec^2 u\ du\\ &= \int_{x=0}^{x=1} \frac{\sec^2 u}{1+\tan^2 u}\ du\\ &= \int_{x=0}^{x=1} \frac{\sec^2 u}{\sec^2 u}\ du\\ &= \int_{x=0}^{x=1} 1\ du\\ &= [u]_{x=0}^{x=1}\\ &= \left[\tan^{-1} x\right]_0^1\\ &= \left(\tan^{-1} 1\right) - \left(\tan^{-1} 0\right)\\ &= \frac{\pi}{4} - 0\\ &= \frac{\pi}{4}\end{aligned}$$

5) Find an integral for which the trapezium method always gives an underestimate. Explain why.

A few examples:

$$\int_0^1 50 - x^2\ dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$$

$$\int_0^1 30 - 30^x\ dx$$

If these integrals are sketched it can be seen that the tops of the trapeziums will always be lower than the curve. This is because the gradients of these curves are decreasing.