MATH6103 Differential & Integral Calculus

MATH6500 Elementary Mathematics for Engineers

Problem Sheet 6 Solutions

1) Find:

a) $\displaystyle\int x\cos x \, dx$

Let $u = x$ and $v' = \cos x$. This means $u' = 1$ and $v = \sin x$. Using integration by parts:

$$\begin{aligned}\int x\cos x \, dx &= uv - \int u'v \, dx \\ &= x\sin x - \int \sin x \, dx \\ &= x\sin x + \cos x + c\end{aligned}$$

b) $\displaystyle\int 5y\ln y \, dy$

Let $u = \ln y$ and $v' = 5y$. This means $u' = \frac{1}{y}$ and $v = \frac{5}{2}y^2$. Using integration by parts:

$$\begin{aligned}\int 5y\ln y \, dy &= uv - \int u'v \, dy \\ &= \frac{5}{2}y^2\ln y - \int \frac{5}{2}y^2 \cdot \frac{1}{y} \, dy \\ &= \frac{5}{2}y^2\ln y - \int \frac{5}{2}y \, dy \\ &= \frac{5}{2}y^2\ln y - \frac{5}{4}y^2 + c\end{aligned}$$

c) $\displaystyle\int_0^1 te^t \, dt$

Let $u = t$ and $v' = e^t$. This means $u' = 1$ and $v = e^t$. Using integration by parts:

$$\begin{aligned}\int_0^1 te^t\ dt &= [uv]_0^1 - \int_0^1 u'v\ dt \\ &= \left[te^t\right]_0^1 - \int_0^1 e^t\ dt \\ &= \left[te^t\right]_0^1 - \left[e^t\right]_0^1 \\ &= \left(1e^1\right) - \left(0e^0\right) - \left(e^1\right) + \left(e^0\right) \\ &= e - 0 - e + 1 \\ &= 1\end{aligned}$$

d) $\displaystyle\int e^x \sin x\ dx$

Let $u = e^x$ and $v' = \sin x$. This means $u' = e^x$ and $v = \cos x$. Using integration by parts:

$$\begin{aligned}\int e^x \sin x\ dx &= [uv]_0^1 - \int_0^1 u'v\ dx \\ &= e^x \cos x - \int e^x \cos x\ dx\end{aligned}$$

Using integration by parts on this integral gives:

$$\int e^x \cos x\ dx = -e^x \sin x + \int e^x \sin x\ dx$$

Therefore:

$$\begin{aligned}\int e^x \sin x\ dx &= e^x \cos x + e^x \sin x - \int e^x \sin x\ dx + c \\ 2\int e^x \sin x\ dx &= e^x \cos x + e^x \sin x + c \\ \int e^x \sin x\ dx &= \frac{e^x \cos x + e^x \sin x}{2} + c\end{aligned}$$

e) $\displaystyle\int \frac{4}{(x+1)(x-2)}\ dx$

First, split the integrand using partial fractions:

$$\begin{aligned}\frac{4}{(x+1)(x-2)} &= \frac{A}{x+1} + \frac{B}{x-2} \\ 4 &= A(x-2) + B(x+1)\end{aligned}$$

$$x = 2 \implies 4 = 3B \implies B = \frac{4}{3}$$

$$x = -1 \implies 4 = -3A \implies A = -\frac{4}{3}$$

Therefore:

$$\int \frac{4}{(x+1)(x-2)}\,dx = \frac{4}{3}\int \frac{1}{x-2}\,dx - \frac{4}{3}\int \frac{1}{x+1}\,dx$$
$$= \frac{4}{3}\ln|x-2| - \frac{4}{3}\ln|x+1| + c$$

f) $\displaystyle\int_1^2 \frac{5x+2}{(x-1)(x+2)}\,dx$

First, split the integrand using partial fractions:

$$\frac{5x+2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$5x+2 = A(x+2) + B(x-1)$$
$$x = 1 \implies 7 = 3A \implies A = \frac{7}{3}$$
$$x = -2 \implies -8 = -3B \implies B = \frac{8}{3}$$

Therefore:

$$\int \frac{5x+2}{(x-1)(x+2)}\,dx = \frac{7}{3}\int \frac{1}{x+2}\,dx + \frac{8}{3}\int \frac{1}{x-1}\,dx$$
$$= \frac{7}{3}\ln|x+2| + \frac{8}{3}\ln|x-1| + c$$

2) Find

$$\int \frac{x}{x^2-1}\,dx$$

by two different methods.

Partial Fractions

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$$
$$= \frac{A}{x-1} + \frac{B}{x+1}$$
$$x = A(x+1) + B(x-1)$$
$$x = 1 \implies 1 = 2A \implies A = \frac{1}{2}$$
$$x = -1 \implies -1 = -2B \implies B = \frac{1}{2}$$

Therefore:

$$\begin{aligned}\int \frac{x}{x^2-1}\ dx &= \frac{1}{2}\int \frac{1}{x+1}\ dx + \frac{1}{2}\int \frac{1}{x-1}\ dx \\ &= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + c\end{aligned}$$

ln **special case**

$$\begin{aligned}\int \frac{x}{x^2-1}\ dx &= \frac{1}{2}\int \frac{2x}{x^2-1}\ dx \\ &= \frac{1}{2}\ln|x^2-1|\end{aligned}$$

Substitution
Let $u = x^2 - 1$.

$$\begin{aligned}\frac{du}{dx} &= 2x \\ \frac{1}{2x}\ du &= \ dx \\ \int \frac{x}{x^2-1}\ dx &= \int \frac{x}{u}\frac{1}{2x}\ du \\ &= \int \frac{1}{2u}\ du \\ &= \frac{1}{2}\ln|2u| + c \\ &= \frac{1}{2}\ln|2(x^2-1)| + c\end{aligned}$$

Bonus Question: Show that the answers obtained from the 3 methods above are all equivalent.