MATH6103 Differential & Integral Calculus

MATH6500 Elementary Mathematics for Engineers

Problem Sheet 3 Solutions

1) Differentiate the following functions:

a) $f(x) = x^2 - x^3$

$$f'(x) = 2x - 3x^2$$

b) $g(x) = 4x^{\frac{1}{2}}$

$$\begin{aligned} g'(x) &= 4 \cdot \frac{1}{2} x^{-\frac{1}{2}} \\ &= \frac{2}{x^{\frac{1}{2}}} \\ &= \frac{2}{\sqrt{x}} \end{aligned}$$

c) $h(x) = \cos x$

$$h'(x) = -\sin x$$

d) $i(x) = \frac{1}{x}$

$$i(x) = x^{-1}$$

$$\begin{aligned} i'(x) &= -x^{-2} \\ &= \frac{-1}{x^2} \end{aligned}$$

e) $j(x) = 359x^{17}$

$$\begin{aligned} j'(x) &= 359 \cdot 17x^16 \\ &= 6103x^16 \end{aligned}$$

f) $k(x) = \sqrt{x}$

$$k(x) = x^{\frac{1}{2}}$$

$$\begin{aligned} k'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2x^{\frac{1}{2}}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

g) $l(x) = \cos^2 x$

$$\begin{aligned} l'(x) &= 2\cos x \cdot \frac{d}{dx}[\cos x] \\ &= 2\cos x \cdot -\sin x \\ &= -2\cos x \sin x \end{aligned}$$

h) $m(x) = \cos x \sin x$

$$\begin{aligned} m'(x) &= \cos x \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[\cos x] \\ &= \cos x \cos x + \sin x \cdot -\sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

i) $n(x) = x \sin x$

$$\begin{aligned} n'(x) &= x \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x] \\ &= x\cos x + \sin x \cdot 1 \\ &= x\cos x + \sin x \end{aligned}$$

j) $o(x) = \sqrt{3x^2 + 8}$

$$o(x) = (3x^2 + 8)^{\frac{1}{2}}$$

$$\begin{aligned} o'(x) &= \frac{1}{2}(3x^2 + 8)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left[3x^2 + 8\right] \\ &= \frac{1}{2}(3x^2 + 8)^{-\frac{1}{2}} \cdot 6x \\ &= \frac{6x}{2\sqrt{3x^2 + 8}} \end{aligned}$$

2) Differentiate the following functions:

a) $p(x) = \cos(x+3)$

$$p'(x) = -\sin(x+3)$$

b) $q(x) = x^3 \sin x$

$$q'(x) = 3x^2 \sin x + x^3 \cos x$$

c) $r(x) = \sin(x^2 - 1)$

$$r'(x) = 2x \cos(x^2 - 1)$$

d) $s(x) = x \sin x \cos x$

$$s'(x) = \sin x \cos x + x \cos^2 x - x \sin^2 x$$

e) $u(x) = \sin^2 x$

$$u'(x) = 2 \sin x \cos x$$

f) $v(x) = \sin^2 x \cos x$

$$v'(x) = 2 \sin x \cos^2 x - \sin^3 x$$

g) $w(x) = \sqrt{\sin x + \cos x}$

$$w'(x) = \frac{\cos x - \sin x}{2\sqrt{\sin x + \cos x}}$$

h) $y(x) = \left(x^{10} - x^2 \sin x\right)^2$

$$y'(x) = 2\left(x^{10} - x^2 \sin x\right)\left(10x^9 - 2x \sin x - x^2 \cos x\right)$$

3) Find the x co-ordinates of the points where $f'(x) = 0$:

a) $f(x) = \sin x$

$$f'(x) = \cos x$$

$\cos x = 0$ when:

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

b) $f(x) = 2x^3 - 15x^2 + 36x - 8$

$$f'(x) = 6x^2 + 30x + 36$$

$6x^2 + 30x + 36 = 0$ when $x^2 + 5x + 6 = 0$ which is when:

$$x = -3 \text{ or } -2$$

c) $f(x) = x^3 - 9x$

$$f'(x) = 3x^2 - 9$$

$3x^2 - 9 = 0$ when:

$$x = \pm\sqrt{3}$$

d) $f(x) = \sin x + x$

$$f'(x) = \cos x + 1$$

$\cos x + 1 = 0$ when $\cos x = -1$ which is when:

$$x = \pm\pi, \pm 3\pi, \pm 5\pi, \pm 7\pi, ...$$

e) $f(x) = \frac{x^4}{4} - \frac{x^3}{3} - 8x^2 + 16x + 42$

$$f'(x) = x^3 - x^2 - 16x + 16$$

$x^3 - x^2 - 16x + 16 = 0$ when:

$$x = -4 \text{ or } 1 \text{ or } 4$$

Challenge: Use the product and chain rules to show that:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{d}{dx}\left[f(x)(g(x))^{-1}\right] \\
&= f(x)\frac{d}{dx}\left[(g(x))^{-1}\right] + (g(x))^{-1}\frac{d}{dx}\left[f(x)\right] \\
&= f(x)\cdot -(g(x))^{-2}\cdot g'(x) + (g(x))^{-1}f'(x) \\
&= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)} \\
&= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)g(x)}{(g(x))^2} \\
&= \frac{-f(x)g'(x) + f'(x)g(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}
\end{aligned}
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