MATH6103 Differential & Integral Calculus

MATH6500 Elementary Mathematics for Engineers

Problem Sheet 2 Solutions

1) Show that:

a) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

Proof: From lectures, we know that:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and

$$\cos(-x) = \cos x$$
$$\sin(-x) = -\sin x$$

Therefore

$$\begin{aligned}\sin(\theta - \phi) &= \sin(\theta + (-\phi)) \\ &= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi) \\ &= \sin\theta\cos\phi - \cos\theta\sin\phi\end{aligned}$$

□

b) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$

Proof: From lectures, we know that:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

and

$$\cos(-x) = \cos x$$
$$\sin(-x) = -\sin x$$

Therefore

$$\begin{aligned}\cos(\theta - \phi) &= \cos(\theta + (-\phi)) \\ &= \cos\theta\cos(-\phi) + \sin\theta\sin(-\phi) \\ &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

□

2) Find the value of $\tan\theta$ when:

a) $\cot\theta = \frac{1}{2}$

$$\tan\theta = \frac{1}{\cot\theta}$$

and so

$$\tan\theta = 2.$$

b) $\theta = \pi^{c}$

$$\tan\pi = 0$$

c) $\sec\theta = 100$ and $\tan\theta$ is negative

$$\begin{aligned}\tan^2\theta + 1 &= \sec^2\theta \\ \tan^2\theta &= \sec^2\theta - 1 \\ &= 100^2 - 1 \\ &= 9999\end{aligned}$$

$\tan\theta$ is negative, so

$$\tan\theta = -\sqrt{9999}.$$

d) $\sin\theta = 0$

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{0}{\cos\theta} \\ &= 0\end{aligned}$$

e) $\sin\theta = \frac{1}{2}$ and $\cos\theta$ is positive

$$\sin^2\theta + \cos^2\theta = 1$$

and so

$$\begin{aligned}\cos\theta &= \sqrt{1 - \sin^2\theta} \\ &= \sqrt{1 - \frac{1}{4}} \\ &= \sqrt{\frac{3}{4}} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{1/2}{\sqrt{3}/2}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

f) $\sin\theta = -\frac{1}{\sqrt{5}}$ and $\cos\theta$ is negative

$$\sin^2\theta + \cos^2\theta = 1$$

and so

$$\begin{aligned}\cos\theta &= -\sqrt{1-\sin^2\theta}\\ &= -\sqrt{1-\frac{1}{25}}\\ &= -\sqrt{\frac{24}{25}}\\ &= -\frac{2\sqrt{6}}{5}\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta}\\ &= \frac{-1/5}{-2\sqrt{6}/5}\\ &= \frac{1}{2\sqrt{6}}\end{aligned}$$

3) Using

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

find $f'(x)$ when:

a) $f(x) = x$

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{x+h-x}{h}\\ &= \lim_{h\to 0}\frac{h}{h}\\ &= \lim_{h\to 0} 1\\ &= 1\end{aligned}$$

b) $f(x) = x + 2$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x + h + 2 - (x+2)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

c) $f(x) = x^2$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x \end{aligned}$$

d) $f(x) = 3x^2$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h)^2 - 3x^2}{h} \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} \\ &= \lim_{h\to 0} \frac{(6xh + 3h^2}{h} \\ &= \lim_{h\to 0} 6x + 3h \\ &= 6x \end{aligned}$$

e) $f(x) = \frac{1}{x+2}$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left(f(x+h) - f(x) \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{x+h+2} - \frac{1}{x+2} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{x+2}{(x+2)(x+h+2)} - \frac{x+h+2}{(x+2)(x+h+2)} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{x+2-x-h-2}{(x+2)(x+h+2)} \right) \\
&= \lim_{h \to 0} \frac{1}{h} \left(\frac{-h}{(x+2)(x+h+2)} \right) \\
&= \lim_{h \to 0} \frac{-1}{(x+2)(x+h+2)} \\
&= \frac{-1}{(x+2)(x+2)} \\
&= \frac{-1}{(x+2)^2}
\end{aligned}
$$

f) **Challenge**: $f(x) = \sin x$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x + h \cos x - \sin x}{h} \\
&= \lim_{h \to 0} \frac{h \cos x}{h} \\
&= \lim_{h \to 0} \cos x \\
&= \cos x
\end{aligned}
$$