MATH6103 Differential & Integral Calculus
MATH6500 Elementary Mathematics for Engineers

Problem Sheet 8 Solutions

1) Solve the following second order ODEs.

* b) $\dfrac{d^2y}{dx^2} - 13\dfrac{dy}{dx} + 12y = 0$

$$\begin{aligned} \frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 12y &= 0 \\ \text{Auxiliary equation is } \lambda^2 - 13\lambda + 12 &= 0 \\ (\lambda - 1)(\lambda - 12) &= 0 \\ \lambda &= 1,\ 12 \\ y &= Ae^x + Be^{12x} \end{aligned}$$

* c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$

$$\begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)^2 &= 0 \\ \lambda &= 1 \\ y &= Ae^x + Bxe^x \end{aligned}$$

* d) $2\dfrac{d^2y}{dx^2} + 28\dfrac{dy}{dx} + 98y = 0$

$$\begin{aligned} 2\frac{d^2y}{dx^2} + 28\frac{dy}{dx} + 98y &= 0 \\ \frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y &= 0 \\ \lambda^2 + 14\lambda + 49 &= 0 \\ (\lambda + 7)^2 &= 0 \\ \lambda &= -7 \\ y &= Ae^{-7x} + Bxe^{-7x} \end{aligned}$$

* e) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = 0$

$$\begin{aligned} \lambda^2 + 6\lambda + 10 &= 0 \\ \lambda &= \frac{-6 \pm \sqrt{36 - 40}}{2} \\ \lambda &= -3 \pm \mathrm{i} \\ y &= e^{-3x}(A\sin x + B\cos x) \end{aligned}$$

* f) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0$

$$\lambda^2 + 8\lambda + 25 = 0$$
$$\lambda = \frac{-8 \pm \sqrt{64 - 100}}{2}$$
$$\lambda = -4 \pm 2\mathrm{i}$$
$$y = e^{-4x}(A\sin 2x + B\cos 2x)$$

2) Solve the following second order ODEs. [Hint: Use your answers from question 1]

* b) $\dfrac{d^2y}{dx^2} - 13\dfrac{dy}{dx} + 12y = 290\sin x$

$$\text{Guess } y = C\sin x + D\cos x$$
$$y' = C\cos x - D\sin x$$
$$y'' = -C\sin x - D\cos x$$

$$\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 12y = 290\sin x$$
$$-C\sin x - D\cos x - 13(C\cos x - D\sin x) + 12(C\sin x + D\cos x) = 290\sin x$$
$$(13D + 11C)\sin x + (11D - 13C)\cos x = 290\sin x$$

$$13D + 11C = 290 \qquad 11D - 13C = 0$$
$$D = \tfrac{13}{11}C$$
$$\frac{169}{11}C + 11C = 290$$
$$\frac{290}{11}C = 290$$
$$C = 11$$
$$D = \frac{13}{11} \times 11$$
$$= 13$$

$$y = Ae^{x} + Be^{12x} + 11\sin x + 13\cos x$$

* c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 2\cos x$

$$\text{Guess } y = C\sin x + D\cos x$$
$$y' = C\cos x - D\sin x$$
$$y'' = -C\sin x - D\cos x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2\cos x$$
$$-2C\cos x + 2D\sin x + = 2\cos x$$

$$-2C = 2 \qquad 2D = 0$$
$$C = 1 \qquad D = 0$$

$$y = Ae^x + Bxe^x + \sin x$$

* d) $2\dfrac{d^2y}{dx^2} + 28\dfrac{dy}{dx} + 98y = 98x - 196$

$$2\frac{d^2y}{dx^2} + 28\frac{dy}{dx} + 98y = 98x - 196$$
$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 49x - 98$$

$$\text{Guess} y = Cx + D$$
$$y' = C$$
$$y'' = 0$$

$$\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 49x - 98$$
$$14C + 49Cx + 49D = 49x - 98$$

$$49C = 49 \qquad 14C + 49D = -98$$
$$C = 1 \qquad 14 + 49D = -98$$
$$C = 1 \qquad D = -\frac{16}{7}$$

$$y = Ae^{-7x} + Bxe^{-7x} + x - \frac{16}{7}$$

* e) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 10y = x^2 - 1$

$$\text{Guess} y = Cx^2 + Dx + E$$
$$y' = 2Cx + D$$
$$y'' = 2C$$

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = x^2 - 1$$
$$2C + 12Cx + 6D + 10Cx^2 + 10Dx + 10E = x^2 - 1$$

$$10C = 1 \qquad 12C + 10D = 0 \qquad 2C + 6D + 10E = -1$$
$$C = \frac{1}{10} \qquad \frac{12}{10} + 10D = 0 \qquad \frac{2}{10} + 6D + 10E = -1$$
$$D = -\frac{12}{100} = -\frac{3}{25} \qquad \frac{20}{100} - \frac{72}{100} + 10E = -1$$
$$E = -\frac{48}{100} = -\frac{12}{25}$$

$$y = e^{-3x}(A\sin x + B\cos x) + \tfrac{1}{10}x^2 - \tfrac{3}{25}x - \tfrac{12}{25}$$

* f) $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 25y = 75$

$$\begin{aligned} \text{Guess } y &= C \\ y' &= 0 \\ y'' &= 0 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y &= 75 \\ 25C &= 75 \\ C &= 3 \end{aligned}$$

$$y = e^{-4x}(A\sin 2x + B\cos 2x) + 3$$