MATH6103 Differential & Integral Calculus
MATH6500 Elementary Mathematics for Engineers

Problem Sheet 8 Solutions

1) Use separation of variables to find the general solutions of the following:

* c) $2x \cos y = (x^2 - 3) \sec y \dfrac{dy}{dx}$ [1 mark]

$$\begin{aligned}
2x \cos y &= (x^2 - 3) \sec y \frac{dy}{dx} \\
\frac{2x}{x^2 - 3}\ dx &= \frac{\sec y}{\cos y}\ dy \\
\frac{2x}{x^2 - 3}\ dx &= \sec^2 y\ dy \\
\int \frac{2x}{x^2 - 3}\ dx &= \int \sec^2 y\ dy \\
\ln(x^2 - 3) + c &= \tan y \\
y &= \tan^{-1}\big(\ln(x^2 - 3) + c\big)
\end{aligned}$$

* d) $\dfrac{dy}{dx} = 2xy$ [1 mark]

$$\begin{aligned}
\frac{dy}{dx} &= 2xy \\
\frac{1}{y}\ dy &= 2x\ dx \\
\int \frac{1}{y}\ dy &= \int 2x\ dx \\
\ln y &= x^2 + c \\
y &= e^{x^2 + c} \\
y &= Ae^{x^2}
\end{aligned}$$

* f) $\dfrac{dx}{dy} + \dfrac{1}{x^2} = 0$ [Notice: $\frac{dx}{dy}$, not $\frac{dy}{dx}$] [1 mark]

$$\begin{aligned}
\frac{dx}{dy} + \frac{1}{x^2} &= 0 \\
\frac{dx}{dy} &= -\frac{1}{x^2} \\
x^2\ dx &= -1\ dy \\
\int x^2\ dx &= \int -1\ dy \\
\frac{x^3}{3} + c &= -y \\
y &= -\frac{x^3}{3} + c
\end{aligned}$$

2) Use integrating factors to find the general solutions of the following:

* b) $\dfrac{dy}{dx} + 2xy = 2x$ [1 mark]

$$\begin{aligned}\text{Integrating factor} &= e^{\int 2x\ dx} \\ &= e^{x^2}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} + 2xy &= 2x \\ e^{x^2}\frac{dy}{dx} + 2e^{x^2}xy &= 2e^{x^2}x \\ \frac{d}{dx}\left(ye^{x^2}\right) &= 2e^{x^2}x \\ ye^{x^2} &= \int 2e^{x^2}x\ dx \\ ye^{x^2} &= e^{x^2} + c \\ y &= 1 + ce^{-x^2}\end{aligned}$$

* d) $\dfrac{dy}{dx} = \cos x - y\tan x$ [1 mark]

$$\begin{aligned}\frac{dy}{dx} &= \cos x - y\tan x \\ \frac{dy}{dx} + y\tan x &= \cos x\end{aligned}$$

$$\begin{aligned}\text{Integrating factor} &= e^{\int \tan x\ dx} \\ &= e^{\ln \sec x} \\ &= \sec x\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} + y\tan x &= \cos x \\ \sec x\frac{dy}{dx} + y\sec x\tan x &= \sec x\cos x \\ \frac{d}{dx}(y\sec x) &= \sec x\cos x \\ y\sec x &= \int \sec x\cos x\ dx \\ &= \int \frac{\cos x}{\cos x}\ dx \\ &= \int 1\ dx \\ &= x + c \\ y &= x\cos x + c\cos x\end{aligned}$$

* e) $x^2\dfrac{dy}{dx} + xy + 1 = 0$ [1 mark]

$$\begin{aligned}
x^2\frac{dy}{dx} + xy + 1 &= 0 \\
\frac{dy}{dx} + \frac{y}{x} &= -\frac{1}{x^2}
\end{aligned}$$

$$\begin{aligned}
\text{Integrating factor} &= e^{\int \frac{1}{x}\ dx} \\
&= e^{\ln x} \\
&= x
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} + \frac{y}{x} &= -\frac{1}{x^2} \\
x\frac{dy}{dx} + y &= -\frac{1}{x} \\
\frac{d}{dx}(xy) &= -\frac{1}{x} \\
xy &= -\int \frac{1}{x}\ dx \\
xy &= -\ln x + c \\
y &= -\frac{1}{x}\ln x + \frac{c}{x}
\end{aligned}$$

3) Solve the following initial value problems:

* c) $\dfrac{dy}{dx} + \dfrac{y}{x} = 6, \quad y(1) = 4$ [1 mark]

$$\begin{aligned}
\text{Integrating factor} &= e^{\int \frac{1}{x}\ dx} \\
&= e^{\ln x} \\
&= x
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} + \frac{y}{x} &= 6 \\
x\frac{dy}{dx} + y &= 6x \\
\frac{d}{dx}(xy) &= 6x \\
xy &= \int 6x\ dx \\
xy &= 3x^2 + c \\
y &= 3x + \frac{c}{x}
\end{aligned}$$

$$\begin{aligned}
y(1) &= 4 \\
3 \cdot 1 + \frac{c}{1} &= 4 \\
3 + c &= 4 \\
c &= 1
\end{aligned}$$

$$y = 3x + \frac{1}{x}$$

* d) $x\frac{dy}{dx} - 3y = 0, \quad y(1) = 5$ [1 mark]

$$\begin{aligned} x\frac{dy}{dx} - 3y &= 0 \\ \frac{1}{y}\ dy &= \frac{3}{x}\ dx \\ \int \frac{1}{y}\ dy &= \int \frac{3}{x}\ dx \\ \ln y &= 3\ln x + c \end{aligned}$$

$$\begin{aligned} y(1) &= 5 \\ \ln 5 &= 3\ln 1 + c \\ \ln 5 &= c \end{aligned}$$

$$\begin{aligned} \ln y &= 3\ln x + \ln 5 \\ \ln y &= \ln 5x^3 \\ y &= 5x^3 \end{aligned}$$

* e) $e^{-x}\frac{dy}{dx} + y = 1, \quad y(0) = 1$ [1 mark]

$$\begin{aligned} e^{-x}\frac{dy}{dx} + y &= 1 \\ \frac{dy}{dx} + e^x y &= e^x \end{aligned}$$

$$\begin{aligned} \text{Integrating factor} &= e^{\int e^x\ dx} \\ &= e^{e^x} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} + e^x y &= e^x \\ e^{e^x}\frac{dy}{dx} + e^{e^x}e^x y &= e^{e^x}e^x \\ \frac{d}{dx}\left(e^{e^x}y\right) &= e^{e^x}e^x \\ e^{e^x}y &= \int e^{e^x}e^x\ dx \\ e^{e^x}y &= e^{e^x} + c \\ y &= 1 + ce^{-e^x} \end{aligned}$$

$$
\begin{aligned}
y(0) &= 1 \\
1 + ce^{e^0} &= 1 \\
1 + ce^1 &= 1 \\
ce &= 0 \\
c &= 0
\end{aligned}
$$

$$y = 1$$

* f) $\dfrac{dy}{dx} = (xy)^2, \quad y(1) = 1$ [1 mark]

$$
\begin{aligned}
\frac{dy}{dx} &= (xy)^2 \\
\frac{dy}{dx} &= x^2y^2 \\
\frac{1}{y^2}\ dy &= x^2\ dx \\
\int y^{-2}\ dy &= \int x^2\ dx \\
-y^{-1} &= \frac{x^3}{3} + c \\
\frac{1}{y} &= -\frac{x^3}{3} + c
\end{aligned}
$$

$$
\begin{aligned}
y(1) &= 1 \\
\frac{1}{1} &= -\frac{1^3}{3} + c \\
1 &= -\frac{1}{3} + c \\
c &= \frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{y} &= -\frac{x^3}{3} + \frac{4}{3} \\
y &= \frac{3}{4 - x^3}
\end{aligned}
$$