MATH6103 Differential & Integral Calculus
MATH6500 Elementary Mathematics for Engineers

Problem Sheet 6 Solutions

1) Use integration by parts to find the following:

* c) $\displaystyle\int \phi e^{\phi}\ d\phi$

$$
\begin{aligned}
u &= \phi & v &= e^{\phi} \\
u' &= 1 & v' &= e^{\phi} \\
\int \phi e^{\phi}\ d\phi &= uv - \int u'v\ d\phi \\
&= \phi e^{\phi} - \int e^{\phi}\ d\phi \\
&= \phi e^{\phi} - e^{\phi} + c
\end{aligned}
$$

* e) $\displaystyle\int_1^e \ln x\ dx$

$$
\begin{aligned}
u &= \ln x & v &= x \\
u' &= \frac{1}{x} & v' &= 1 \\
\int_1^e \ln x\ dx &= [uv]_1^e - \int_1^e u'v\ dx \\
&= [x \ln x]_1^e - \int_1^e 1\ dx \\
&= [x \ln x]_1^e - [x]_1^e \\
&= e \ln e - 1 \ln 1 - e + 1 \\
&= e - e + 1 \\
&= 1
\end{aligned}
$$

* f) $\int e^x \sin x \; dx$

$$u = e^x \qquad v = -\cos x$$
$$u' = e^x \qquad v' = \sin x$$
$$\int e^x \sin x \; dx = -e^x \cos x + \int e^x \cos x \; dx$$
$$u = e^x \qquad v = \sin x$$
$$u' = e^x \qquad v' = \cos x$$
$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \; dx$$
$$\text{Let } I = \int e^x \sin x \; dx$$
$$I = -e^x \cos x + e^x \sin x - I$$
$$2I = -e^x \cos x + e^x \sin x$$
$$I = \frac{e^x \sin x - e^x \cos x}{2} + c$$

2) Use partial fractions to find the following:

* a) $\int \frac{1}{x^2 + 7x + 6} \; dx$

$$\frac{1}{x^2+7x+6} = \frac{1}{(x+6)(x+1)}$$
$$= \frac{A}{x+6} + \frac{B}{x+1}$$
$$1 = A(x+1) + B(x+6)$$

$$x = -1 \qquad \Longrightarrow \qquad 1 = 5B$$
$$B = \frac{1}{5}$$
$$x = -6 \qquad \Longrightarrow \qquad 1 = -5A$$
$$A = -\frac{1}{5}$$

$$\frac{1}{x^2+7x+6} = \frac{1}{5} \cdot \frac{1}{x+1} - \frac{1}{5} \cdot \frac{1}{x+6}$$
$$\int \frac{1}{x^2+7x+6} \; dx = \frac{1}{5} \int \frac{1}{x+1} \; dx - \frac{1}{5} \int \frac{1}{x+6} \; dx$$
$$= \frac{1}{5} \ln(x+1) - \frac{1}{5} \ln(x+6) + c$$

* c) $\int_0^1 \frac{x-1}{(x+1)(x^2+1)} \; dx$

$$\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$
$$x - 1 = A(x^2+1) + (Bx+C)(x+1)$$

$$
\begin{aligned}
x = -1 \quad &\Longrightarrow \quad -2 = 2A \\
&\qquad\quad A = -1 \\
x = 0 \quad &\Longrightarrow \quad -1 = A + C \\
&\qquad\quad -1 = -1 + C \\
&\qquad\quad C = 0 \\
x = 1 \quad &\Longrightarrow \quad 0 = 2A + 2(B + C) \\
&\qquad\quad 0 = -2 + 2B \\
&\qquad\quad B = 1
\end{aligned}
$$

$$
\begin{aligned}
\frac{x-1}{(x+1)(x^2+1)} &= \frac{-1}{x+1} + \frac{x}{x^2+1} \\
\int_0^1 \frac{x-1}{(x+1)(x^2+1)}\ dx &= \int_0^1 \frac{-1}{x+1}\ dx + \int_0^1 \frac{x}{x^2+1}\ dx \\
&= [-\ln(x+1)]_0^1 + \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 \\
&= (-\ln 2) - (-\ln 1) + \frac{1}{2}(\ln 2) - \frac{1}{2}(\ln 1) \\
&= -\ln 2 + \frac{1}{2}\ln 2 \\
&= -\frac{1}{2}\ln 2
\end{aligned}
$$

3) Find the following:

* b) $\displaystyle\int \frac{\ln b}{b}\ db$

$$
\begin{aligned}
\text{Let } u &= \ln b \\
du &= \frac{1}{b}\ db \\
\int \frac{\ln b}{b}\ db &= \int u\ du \\
&= \frac{1}{2}u^2 + c \\
&= \frac{1}{2}(\ln b)^2 + c
\end{aligned}
$$

* c) $\displaystyle\int (x-3)\sin(x^2-6x+12)\ dx$

$$\begin{aligned}
\text{Let } u &= x^2-6x+12\\
du &= 2x-6\ dx\\
\frac{1}{2}\ du &= x-3\ dx\\
\int (x-3)\sin(x^2-6x+12)\ dx &= \frac{1}{2}\int \sin u\ du\\
&= -\frac{1}{2}\cos u + c
\end{aligned}$$

* d) $\displaystyle\int_0^1 y^e + e^y\ dy$

$$\begin{aligned}
\int_0^1 y^e + e^y\ dy &= \left[\frac{y^{e+1}}{e+1} + e^y\right]_0^1\\
&= \left(\frac{1^{e+1}}{e+1} + e^1\right) - \left(\frac{0^{e+1}}{e+1} + e^0\right)\\
&= \frac{1}{e+1} + e - 1\\
&= \frac{e^2}{e+1}
\end{aligned}$$

* 4a) Use the trapezium rule with $h = \frac{1}{2}$ to approximate the following integral. You may use a calculator or spreadsheet.

$$\int_0^3 e^{x^2}\ dx$$

x	0	0.5	1	1.5	2	2.5	3
e^{x^2}	1	1.28	2.72	9.49	54.60	518.01	8103.08

$$\begin{aligned}
\int_0^3 e^{x^2}\ dx &\approx \frac{1}{2}\cdot\frac{1}{2}(1+8103.08) + \frac{1}{2}(1.28+2.72+9.49+54.60+518.01)\\
&= 2319.07
\end{aligned}$$

* b) Give one suggestion of what you could do to improve the accuracy of your answer to (4a).
Make h smaller / Use more trapeziums