2.6 Exponentials & Logarithms

An exponential function is a function of the form

$$f(x) = a^x,$$

where a is a positive constant.

Example

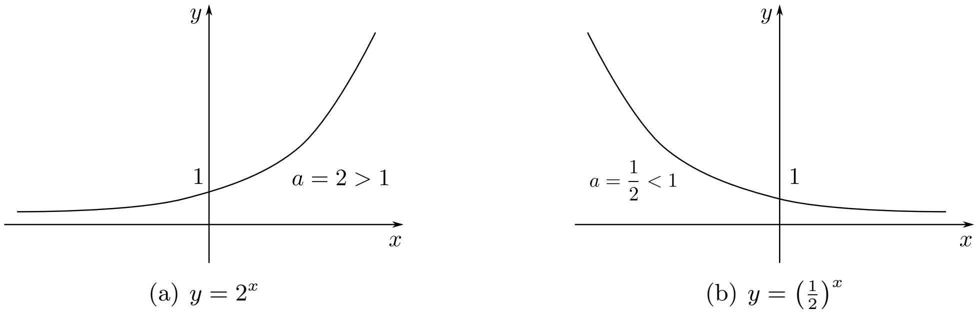


Figure 2.8: Two exponential functions

Properties

If $a > 1$, $f(x)$ increases as x increases.

If $a < 1$, $f(x)$ decreases as x increases.

If $a = 1$, $f(x) = 1$.

$a^0 = 1$ for each a, so the graph always passes through the point $(0, 1)$.

2.6.1 The gradient of e^x

First let us consider the gradient of a^x at $x = 0$.

Example
Suppose we have $f(x) = 2^x$, then applying the definition of the derivative we have

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{2^h - 1}{h}.$$

h	$f'(0) \approx$
0.1	0.7177
0.01	0.6955
0.001	0.6933
0.0001	0.6932

So for $f(x) = 2^x$, $(a = 2)$, we have slope ≈ 0.693 at $x = 0$.

Similarly, for $f(x) = 3^x$, $(a = 3)$, we have slope ≈ 1.698 at $x = 0$.

Therefore, we expect that there is a number between 2 and 3 such that the slope at $x = 0$ is 1. This number is called e, where $e \approx 2.718281828459\ldots$. The number e is irrational.

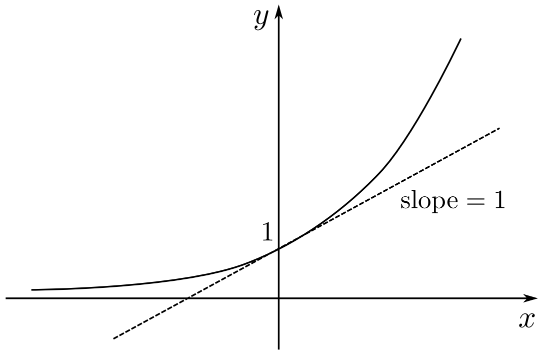


Figure 2.9: *Graph of $y = e^x$, which has a gradient of 1 at $x = 0$.*

The fact that the slope is 1 at $x = 0$ tells us that

$$\frac{e^h - e^0}{h} = \frac{e^h - 1}{h} \to 1, \quad \text{as } h \to 0.$$

Definition
We call $f(x) = e^x = \exp(x)$ the **exponential function**.

Property

$$\frac{d}{dx}(e^x) = e^x.$$

Proof: To find the gradient at $x = c$, we need to look at

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{e^{c+h} - e^c}{h} \\ &= \lim_{h\to 0} \frac{e^c(e^h - 1)}{h} \\ &= e^c \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^c, \end{aligned}$$

□

Example
To find

$$\frac{d}{dx}\left(e^{\sqrt{1+x}}\right)$$

we must use the chain rule. Choose $g(x) = \sqrt{1+x}$ and $f(u) = e^u$, so we have $g'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ and $f'(u) = e^u$.

$$\begin{aligned}\frac{d}{dx}\left(e^{\sqrt{1+x}}\right) &= f'(g(x))g'(x)\\ &= e^{\sqrt{1+x}} \cdot \frac{1}{2}(1+x)^{-\frac{1}{2}}\\ &= \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}}.\end{aligned}$$

2.6.2 Logarithms

Definition
The inverse of an exponenetial function is called a **logarithm** or **log**.
The inverse of a^x is written as $\log_a x$ and is defined such that

$$a^{\log_a x} = x.$$

Example
$\log_{10}(1000) = 3$ because $10^3 = 1000$.
$\log_2(16) = 4$ because $2^4 = 16$.
$\log_{10}(2) = 0.301...$ because $10^{0.301...} = 2$.

Laws of Logs
The following properties hold for logarithms:

1. $\log_a(MN) = \log_a M + \log_a N$.
2. $\log_a(M^p) = p\log_a M$.

Logarithms are used among other things to solve exponential equations.

Example
Find x, given $3^x = 7$. Taking the logarithm of both sides we have

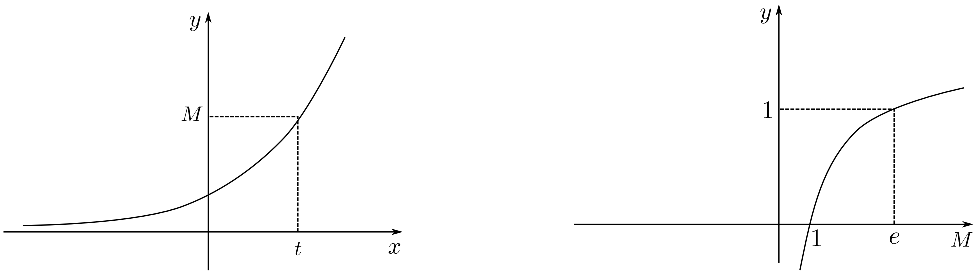
$$\ln(3^x) = \ln 7 \qquad \Longrightarrow \qquad x\ln 3 = \ln 7.$$

Rearranging we have

$$x = \frac{\ln 3}{\ln 7} \approx \frac{1.95}{1.10} \approx 1.77.$$

The natural logarithm

Definition
The inverse of $f(x) = e^x$ is called the **natural logarithm** and is written $\ln x$.



(a) $y = e^x$.

(b) $y = \ln x$.

Figure 2.10: Graphs of the exponential functions and the natural logarithm.

2.6.3 Differentiation of other exponentials

Example
In order to differentiate 3^x, we must express it in terms of e^x:

$$3 = e^{\ln 3} \quad \Longrightarrow \quad 3^x = (e^{\ln 3})^x = e^{x \ln 3}$$

Therefore we calculate the derivative of 3^x as follows:

$$\begin{aligned} \frac{d}{dx}(3^x) &= \frac{d}{dx}(e^{x \ln 3}) \\ &= e^{x \ln 3} \cdot \ln 3 \\ &= 3^x \cdot \ln 3. \end{aligned}$$

In general, for any positive constant a

$$\frac{d}{dx}(a^x) = a^x \ln a.$$