p is an exponential function

If p is an exponential function then we guess that the particular integral will be an exponential function.

Example
Find the general solution of the following differential equation,

$$y'' + 4y' + 3y = 5e^{4x}.$$

The auxiliary equation is

$$\lambda^2 + 4\lambda + 3 = 0 \quad \Longleftrightarrow \quad (\lambda + 1)(\lambda + 3) = 0.$$

Hence, this has two distinct real roots, namely $\lambda_1 = -1$, $\lambda_2 = -3$. So the C.F. (from the homogeneous equation) is given by

$$g(x) = c_1e^{-x} + c_2e^{-3x}.$$

To find the P.I. we try

$$f = ae^{4x},$$

since $p(x) = 5e^4x$. Differentiating, we have

$$f' = 4a^{4x}, \quad f'' = 16a^{4x}.$$

Substituting into the differential equation we see that

$$\begin{aligned} f'' + 4f' + 3f &= 16ae^{4x} + 16ae^{4x} + 3ae^{4x} \\ &= 35ae^{4x} \\ &= 35ae^{4x} \qquad\qquad \equiv 5e^{4x}. \end{aligned}$$

Therefore we have $a = 1/7$ and so the P.I. is

$$f = \frac{1}{7}e^{4x}.$$

Finally, the general solution is

$$y(x) = \frac{1}{7}e^{4x} + c_1e^{-x} + c_2e^{-3x}.$$

Example
Solve the initial-value problem given by

$$y'' + 4y' + 3y = e^{-x}, \quad y(0) = 0, \quad y'(0) = 0.$$

We know from the example above that the C.F. is

$$c_1e^{-x} + c_2e^{-3x},$$

Now let us find the P.I., if we try $f = ae^{-x}$, we know that it wouldn't work since ae^{-x} is actually a solution to the homogeneous equation $y''+4y'+3y = 0$. Therefore, we try

$$f = axe^{-x},$$

thus

$$f' = ae^{-x} - axe^{-x}, \quad \text{and} \quad f'' = -2ae^{-x} + axe^{-x}.$$

Substituting into the differential equation we have

$$\begin{aligned} f'' + 4f' + 3f &= -2ae^{-x} + axe^{-x} + 4ae^{-x} - 4axe^{-x} + 3axe^{-x} \\ &= 2ae^{-x} \\ &\equiv e^{-x}. \end{aligned}$$

Therefore we must have $a = 1/2$ and so the general solution to the differential equations is

$$y(x) = \frac{1}{2}xe^{-x} + c_1e^{-x} + c_2e^{-3x}.$$

Differentiating the general solution we have

$$y'(x) = \frac{1}{2}e^{-x} - \frac{1}{2}xe^{-x} - c_1e^{-x} - 3c_2e^{-3x}.$$

Putting $x = 0$, we have (from the initial conditions)

$$\left.\begin{array}{r} y(0) = c_1 + c_2 = 0 \\ y'(0) = \frac{1}{2} - c_1 - 3c_2 = 0 \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{l} c_1 = -\frac{1}{4} \\ c_2 = \frac{1}{4} \end{array},$$

thus

$$y(x) = \frac{1}{2}xe^{-x} - \frac{1}{4}e^{-x} + \frac{1}{4}e^{-3x},$$

is the solution to the initial-value problem.

Example
Find the general solution to the equation

$$y'' + 2y' + y = 2e^{-x}.$$

Find the C.F.: The auxiliary equation is

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda + 1)^2 &= 0 \\ \lambda_1 &= -1 \end{aligned}$$

We have a repeated root so

$$g(x) = c_1e^{-x} + c_2xe^{-x}.$$

Here e^{-x} and xe^{-x} are two independent solutions to the homogeneous equation

$$y'' + 2y' + y = 0.$$

Find a P.I.: We have to try

$$f = ax^2e^{-x},$$

since e^{-x} and xe^{-x} can't be the solution to the original differential equation as they satisfy the homogeneous equation. The derivatives are:

$$f' = 2axe^{-x} - ax^2e^{-x}$$

$$\begin{aligned} f'' &= 2ae^{-x} - 2axe^{-x} - 2axe^{-x} + ax^2e^{-x} \\ &= axe^{-x} - 4axe^{-x} + 2ae^{-x} \end{aligned}$$

Substituting $y = f(x)$ into the differential equation, we have:

$$\begin{aligned} f'' + 2f' + f &= ax^2e^{-x} - 4axe^{-x} + 2ae^{-x} + 4axe^{-x} - 2ax^2e^{-x} + ax^2e^{-x} \\ &= 2ae^{-x} \\ 2ae^{-x} &\equiv 2e^{-x} \end{aligned}$$

Therefore $a = 1$. So finally, we have the general solution

$$y(x) = (c_1 + c_2x + x^2)e^{-x}.$$