3.3.6 Partial fractions

We can sometimes split polynomial fractions into smaller parts.

Example

$$\frac{1}{x^2-1}$$

First factorise the denominator:

$$x^2-1=(x+1)(x-1)$$

Now write:

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1},$$

where A and B are constants to be found. Multiplying everything by $(x-1)(x+1)$ gives:

$$1=A(x+1)+B(x-1)$$

Substituting in $x=1$ gives

$$1=2A.$$

Substituting in $x=-1$ gives

$$1=-2B.$$

Therefore,

$$\begin{aligned} A &= \frac{1}{2}, \\ B &= -\frac{1}{2}. \end{aligned}$$

Hence:

$$\frac{1}{x^2-1}=\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$$

Definition
Splitting the fraction as above is called splitting into **partial fractions**.

We can use partial fractions for integration:

Example

$$\begin{aligned} \int \frac{1}{x^2-1}\,dx &= \frac{1}{2}\int \frac{1}{x-1}\,dx-\frac{1}{2}\int \frac{1}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+c \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c. \end{aligned}$$

Example
Consider the integral

$$\int \frac{x^2+6x+1}{3x^2+5x-2}\,dx.$$

To split into partial fractions, the numerator needs to be of a lower degree than the denominator. So we manipulate as follows:

$$\begin{aligned}\frac{x^2+6x+1}{3x^2+5x-2} &= \frac{\frac{1}{3}(3x^2+18x+3)}{3x^2+5x-2}\\ &= \frac{\frac{1}{3}(3x^2+5x-2+13x+5)}{3x^2+5x-2}\\ &= \frac{1}{3}\left[1+\frac{13x+5}{3x^2+5x-2}\right].\end{aligned}$$

Now we factorise the denominator:

$$3x^2+5x-2=(3x-1)(x+2)$$

Next we write split the fraction into partial fractions

$$\frac{13x+5}{3x^2+5x-2}=\frac{A}{3x-1}+\frac{B}{x+2}=\frac{(A+3B)x+2A-B}{(3x-1)(x+2)}.$$

Matching the coefficients on the numerator we must have

$$\begin{aligned}A+3B&=13\\ 2A-B&=5\end{aligned}$$

Solving simultaeneously we have the solution $A=4$ and $B=3$. So the integral becomes

$$\begin{aligned}\int \frac{x^2+6x+1}{3x^2+5x-2}\,dx &= \frac{1}{2}\int 1+\frac{13x+5}{3x^2+5x-2}\,dx\\ &= \frac{1}{3}\int dx+\frac{1}{3}\int\frac{4}{3x-1}\,dx+\frac{1}{3}\int\frac{3}{x+2}\,dx\\ &= \frac{1}{3}x+\frac{4}{9}\ln|3x-1|+\ln|x+2|+c.\end{aligned}$$