3.3.2 A special case

Let us consider the derivative of the logarithm of some general function $f(x)$,:

$$\begin{aligned}\frac{d}{dx}\left(\ln(f(x))\right) &= \frac{1}{f(x)}\cdot\frac{d}{dx}\left(f(x)\right)\\ &= \frac{f'(x)}{f(x)}\end{aligned}$$

This implies that:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c$$

Example
Consider the the following integral:

$$I = \int \frac{2x+5}{x^2+5x+3}\,dx.$$

Now, if we choose $f(x) = x^2 + 5x + 3$, then $f'(x) = 2x + 5$. So, if we differentiate $\ln(f(x))$, in this case we have

$$\frac{d}{dx}\left[\ln(x^2+5x+3)\right] = \frac{2x+5}{x^2+5x+3},$$

by the chain rule. Thus, we know the integral must be

$$I = \ln(x^2+5x+3) + c.$$

Example
Consider the following integral:

$$I = \int \frac{3}{2x+2}\,dx.$$

Now, if we choose $f(x) = 2x + 2$ then $f'(x) = 2$. However, the numerator of the integrand is 3. Not to worry, as we can simply re-write or manipulate the initial integral as follows:

$$I = \int \frac{3}{2x+2}\,dx = 3\int \frac{1}{2}\frac{2}{2x+2}\,dx = \frac{3}{2}\int \frac{2}{2x+2}\,dx.$$

Since 3/2 is a constant, which we are able to take out of the integral sign, we need not worry about this and can proceed with the integration using what we have learnt above, giving

$$I = \frac{3}{2}\ln(2x+2) + c.$$

To check, we differentiate the above expression, so

$$\frac{dI}{dx} = \frac{d}{dx}\left[\frac{3}{2}\ln(2x+2) + c\right] = \frac{3}{2}\cdot\frac{1}{2x+2}\cdot 2,$$

which is correct!

This "special case" is an example of a method called *substitution*, and is not limited to integrals which give you logarithms.

3.3.3 Substitution

We can use substitution to convert a complicated integral into a simple one.

Example
We want to find
$$\int (2x+3)^{100}\ dx.$$
We make the substitution
$$u = 2x+3.$$
This means that
$$\frac{du}{dx} = 2$$
and so
$$dx = \frac{1}{2} du.$$
So we calculate the integral as follows:
$$\begin{aligned}\int (2x+3)^{100}\ dx &= \int u^{100} \cdot \frac{1}{2}\ du \\ &= \frac{1}{2}\int u^{100}\ du \\ &= \frac{1}{2} \cdot \frac{1}{101} u^{101} \\ &= \frac{1}{202}(2x+3)^{101} + c.\end{aligned}$$
We can check the result by performing the following differentiation:
$$\frac{d}{dx}\left[\frac{1}{202}(2x+3)^{101} + c\right] = \frac{101}{202}(2x+3)^{100} \cdot 2 = (2x+3)^{100},$$
which is correct.

$\frac{dx}{du}$ is not really a fraction so cannot really be split up as we did here. What we are actually doing is using the following:

Integration by Substitution
$$\int_a^b f(u(x))\ dx = \int_{u(a)}^{u(b)} f(u)\frac{dx}{du}\ du.$$

Substituting the u and $\frac{dx}{du}$ into this formula is equivalent to the splitting up of $\frac{dx}{du}$ which we did. The splitting method gives the correct answer and can be thought of as an easier way to remember the method we are actually using.

Example

$$\int x(x+1)^{50}\ dx$$

Try the substitution $u = x + 1$ (i.e. $x = u - 1$). This gives:

$$\frac{du}{dx} = 1$$

$$dx = du$$

So we have

$$\begin{aligned}
\int x(x+1)^{50}\ dx &= \int (u-1)u^{50}\ du \\
&= \int u^{51}\ du - \int u^{50}\ du \\
&= \frac{1}{52}u^{52} - \frac{1}{51}u^{51} + c \\
&= \frac{1}{52}(x+1)^{52} - \frac{1}{51}(x+1)^{51} + c
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\frac{1}{52}(x+1)^{52} - \frac{1}{51}(x+1)^{51} + c\right) = (x+1)^{51} - (x+1)^{50} = (x+1)^{50} \cdot x,$$

which is correct.

Example

$$\int \frac{1}{x \ln x}\ dx$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$dx = x\ du$$

$$\begin{aligned}
\int \frac{1}{x \ln x}\ dx &= \int \frac{1}{xu} \cdot x\ du \\
&= \int \frac{1}{u}\ du \\
&= \ln|u| + c \\
&= \ln|\ln x| + c.
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\ln|\ln x|\right) = \frac{1}{\ln x} \cdot \frac{1}{x},$$

which is correct.

Example

$$\int \frac{1}{1+\sqrt{x}}\ dx$$

Let $u = 1+\sqrt{x}$.

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$dx = 2x^{\frac{1}{2}}\ du = 2(u-1)\ du$$

$$\begin{aligned}
\int \frac{1}{1+\sqrt{x}}\ dx &= \int \frac{1}{u}\cdot 2(u-1)\ du \\
&= 2\int \left(1-\frac{1}{u}\right)\ du \\
&= 2\int\ du - 2\int \frac{1}{u}\ du \\
&= 2u - 2\ln|u| + c \\
&= 2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + c.
\end{aligned}$$

Check:

$$\frac{d}{dx}\big(2(1+\sqrt{x}) - 2\ln|1+\sqrt{x}| + c\big) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+\sqrt{x})} = \frac{1+\sqrt{x}-1}{\sqrt{x}(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}.$$

Example

$$\int \sin(3x+1)\ dx$$

Let $u = 3x+1$.

$$\frac{du}{dx} = 3$$

$$dx = \frac{1}{3}[u]$$

$$\begin{aligned}
\int \sin 3x+1\ dx &= \frac{1}{3}\int \sin u\ du \\
&= -\frac{1}{3}\cos u + c \\
&= -\frac{1}{3}\cos(3x+1) + c.
\end{aligned}$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{3}\cos(3x+1)+c\right) = +\frac{1}{3}\cdot 3\cdot\sin(3x+1) = \sin(3x+1).$$

Example

$$\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx$$

Let $u = \frac{1}{x}$.

$$\frac{du}{dx} = -\frac{1}{x^2}$$
$$dx = -x^2\, du$$

$$\begin{aligned}\int \frac{\sin\left(\frac{1}{x}\right)}{x^2}\, dx &= \int \frac{\sin u}{x^2} \cdot (-x^2)\, du \\ &= -\int \sin u\, du \\ &= \cos u + c \\ &= \cos\left(\frac{1}{x}\right) + c.\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\cos\left(\frac{1}{x}\right) + c\right) = -\sin\left(\frac{1}{x}\right) \cdot \left(-x^{-2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2},$$

which is correct.