Chapter 1

Functions

1.1 What is a function?

A function takes an input, does something to it, then gives an output.

Example
If the function "add two" is given 4 as an input it would give 6 as its output.
We will represent this function as $f(x) = x + 2$.

1.2 Domains, ranges and variables

In this section, we will look at some properties of functions. First we define some notation we will use when writing sets.

Definition
A **set** is a collection of objects, often numbers. An object in a set is called an element of the set.
Sets are written as a list of items inside curly braces: $\{$ and $\}$.
If an object x is in a set A, we will write $x \in A$. If an object y is not in a set A, we will write $x \notin A$.

Example
The following are examples of sets:

(i) $A = \{1, 2, 8\}$ is a set containing the numbers 1, 2 and 8. We could write $8 \in A$ and $4 \notin A$;

(ii) $\mathbb{Z} = \{\text{all whole numbers}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$;

(iii) $\mathbb{N} = \{\text{all positive whole numbers}\} = \{0, 1, 2, 3, \dots\}$;

(iv) $\mathbb{R} = \{\text{all real numbers}\}$.

Definition
If we have two sets A and B, then a **function** $f : A \to B$ is a rule that sends each element x in A to exactly one element called $f(x)$ in B.
We call the set A the **domain** of f. The **range** of f is the set of all values which f gives as an output (B or a subset of B).
If $x \in A$ then x is called an **independent variable**. If y is in the range of f then y is called a **dependent variable**.

1.3 Representing a function

Functions are sometimes represented by mapping diagrams:

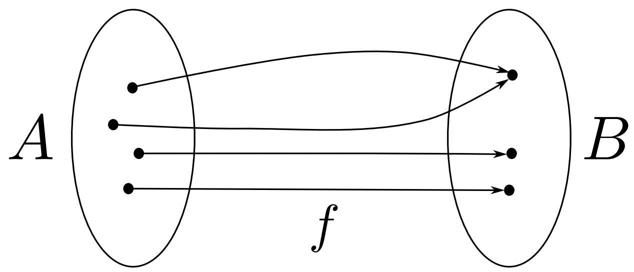


Figure 1.1: *Function f 'maps' elements from set A on to elements in set B. Many-to-one relationship.*

The mapping diagram shows the domain and range of the function. The arrows show where is member of A is sent by the function.

Definition
Let $f : A \to B$ be a function.
If, whenever x and y are in A and not equal, $f(x)$ and $f(y)$ are not equal, the function is called **one-to-one**.
If there are an x and a y in A which are not equal, but $f(x)$ and $f(y)$ are equal, the function is called **many-to-one**.

Definition
Let $f : A \to B$ be a function.
If $f(-x) = f(x)$, f is called an **even** function.
If $f(-x) = -f(x)$, f is called an **odd** function.

Definition
If $f(x + T) = f(x)$ for all x, then we say that $f(x)$ is a **periodic function** with

period T.

Definition
A function, f, has a **vertical asymptote** at $x = a$ if as $x \to a$, $f(x) \to \pm\infty$.
A function, f, has a **horizontal asymptote** at $y = b$ if as $x \to \infty$, $f(x) \to b$ or if as $x \to -\infty$, $f(x) \to b$.

1.4 Polynomials

Definition
A polynomial is a function P with a general form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \tag{1.1}$$

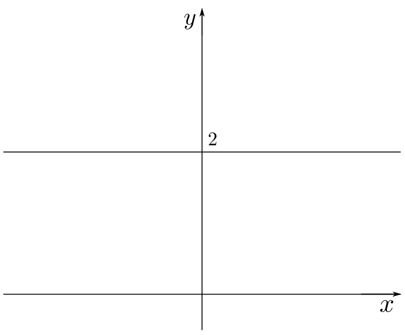
where the coefficients a_i $(i = 0, 1, \ldots, n)$ are numbers and n is a non-negative whole number. The highest power whose coefficient is not zero is called the **degree** of the polynomial.

Example

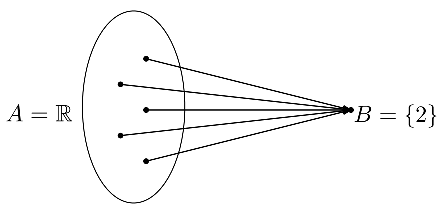
$P(x)$	2	$3x^2 + 4x + 2$	$\frac{1}{1+x}$	$\sqrt{x}$	$1 - 3x + \pi x^3$	$2t + 4$
Polynomial?	Yes	Yes	No	No	Yes	Yes
Order	0	2	N/A	N/A	3	1

1.4.1 Degree 0 polynomials

$P_0(x) = a_0 x^0 = a_0$, say $P(x) = 2$. This polynomial is simply a constant. Degree 0 polynomials are not very interesting.



(a) $y = P(x) = 2$.



(b) *Entire domain mapped to one point.*

Figure 1.2: *A degree 0 polynomial.*

1.4.2 Degree 1 polynomials

$P_1(x) = a_0x^0 + a_1x^1 = ax + b$ $(a \neq 0)$. These are called *linear*, since the graph of $y = ax + b$ is a straight line. The linear equation $ax + b = 0$ has solution $x = -b/a$.

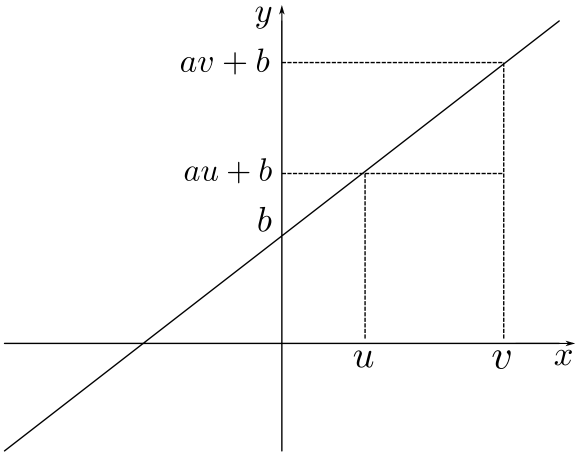


Figure 1.3: *Linear graph given by $y = ax + b$.*

The **gradient** of $y = ax + b$ is a. It can be worked out as follows:

$$\begin{aligned} \text{gradient} &= \frac{\text{change in height}}{\text{change in distance}} \\ &= \frac{\text{change in } y}{\text{change in } x} \\ &= \frac{(av+b)-(au+b)}{v-u} \\ &= \frac{a(v-u)}{v-u} = a. \end{aligned} \tag{1.2}$$

1.4.3 Degree 2 polynomials

$P_2(x) = a_0x^0 + a_1x^1 + a_2x^2 = ax^2 + bx + c$, $a \neq 0$. This is known as a quadratic polynomial. There are three common ways to solve the quadratic equation $ax^2 + bx + c = 0$:

1. Factorising

Some (but not all) quadratics can be factorised.

To factorise $x^2 + bx + c$, look for r_1 and r_2 such that:

- $r_1r_2 = c$
- $r_1 + r_2 = b$

Then $x^2 + bx + c = (x + r_1)(x + r_2)$

Proof: Expanding the brackets in $(x+r_1)(x+r_2)$ gives:

$$x^2 + (r_1 + r_2)x + r_1 r_2.$$

Setting this equal to $x^2 + bx + c$ gives the conditions above.

$\square$

Example
Let $P(x) = x^2 - 3x + 2$.
P can be factorised as $P(x) = (x-2)(x-1)$.
The solutions of $P(x) = 0$ are $x = 2$ and $x = 1$.
note: In this example, b is negative.

2. Completing the square

Completing the square is best demonstrated with an example:

Example
Let $P(x) = x^2 - 3x + 2$ ($b = -3$,$c = 2$).
First, add and subtract $\left(\frac{b}{2}\right)^2$ from $P(x)$:

$$P(x) = x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 + 2$$

This has not changed the value of P as we have added and subtracted the same thing, but it allows us to factorise the first three terms, giving:

$$P(x) = \left(x - \frac{3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 + 2$$

We have completed the square. If we now need to solve $P(x) = 0$:

$$\left(x - \frac{3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 + 2 = 0 \tag{1.3}$$

$$\left(x - \frac{3}{2}\right)^2 = \left(\frac{-3}{2}\right)^2 - 2 \tag{1.4}$$

$$= \frac{1}{4} \tag{1.5}$$

$$x - \frac{3}{2} = \pm\frac{1}{2} \tag{1.6}$$

$$x = \frac{3}{2} \pm \frac{1}{2} \tag{1.7}$$

$$x = 1 \text{ or } 2 \tag{1.8}$$

3. The quadratic formula

Completing the square then with a general quadratic equation $ax^2 + bx + c = 0$ gives the quadratic formula for the solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.9}$$

Sketching a quadratic

The graph of $P(x) = x^2 - 3x + 2$ looks like:

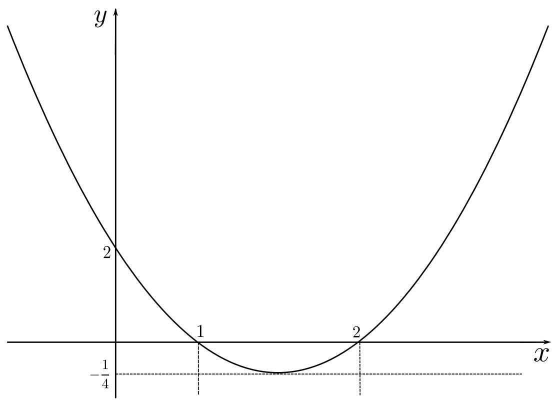


Figure 1.4: *Quadratic graph given by $y = x^2 - 3x + 2$.*

We know (from above):

(i) the graph is a "cup" rather than a "cap" since the coefficient of x^2 is positive. Also we can easily see $P(0) = 2$;

(ii) $P(x) = 0$ at $x = 1$ and $x = 2$;

(iii) $P(x)$ is minimal at $x = \frac{3}{2}$ and $P\left(\frac{3}{2}\right) = -\frac{1}{4}$. Note,

$$\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4},$$

since anything squared is always positive!