MATH6103 Differential & Integral Calculus

MATH6500 Elementary Mathematics for Engineers

Problem Sheet 8 Solutions

1) Use seperation of variables to solve the following:

a) $\dfrac{dy}{dx} + \dfrac{k}{x^2} = 0$

$$\begin{aligned} \frac{dy}{dx} + \frac{k}{x^2} &= 0 \\ \frac{dy}{dx} &= -\frac{k}{x^2} \\ \int dy &= -k \int x^{-2}\ dx \\ y &= kx^{-1} + c \\ y &= \frac{k}{x} + c \end{aligned}$$

b) $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} \\ \int \frac{1}{y}\ dy &= \int \frac{1}{x}\ dx \\ \ln y &= \ln x + c \\ y &= Ax \end{aligned}$$

c) $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y} \\ \int y\ dy &= \int x\ dx \\ y^2 &= x^2 + c \end{aligned}$$

d) $\dfrac{dx}{dy} + \dfrac{k}{x^2} = 0$

$$\frac{dx}{dy} + \frac{k}{x^2} = 0$$
$$\frac{dx}{dy} = -\frac{k}{x^2}$$
$$\int x^2 \ dx = \int -k \ dy$$
$$\frac{1}{3}x^3 + c = -ky$$
$$y = -\frac{1}{3k}x^3 - c$$

e) $\sin y \cos x \frac{dy}{dx} = \sin x \cos y$

$$\sin y \cos x \frac{dy}{dx} = \sin x \cos y$$
$$\int \tan y \ dy = \int \tan x \ dx$$
$$-\ln |\cos y| = -ln|\cos x| + c$$
$$\ln |\cos y| = ln|\cos x| - c$$
$$\cos y = A \cos x$$

f) $\frac{1+y^2}{(1+x^2)xy} = \frac{dy}{dx}$

$$\frac{1+y^2}{(1+x^2)xy} = \frac{dy}{dx}$$
$$\int \frac{1}{(1+x^2)x} \ dx = \int \frac{y}{1+y^2} \ dy$$
$$\frac{1}{2}\ln|x^2| - \frac{1}{2}\ln|1+x^2| + c = \frac{1}{2}\ln|1+y^2|$$
$$A\frac{x^2}{1+x^2} = 1 + y^2$$

g) $xy\frac{dy}{dx} = x$

$$\begin{aligned} xy\frac{dy}{dx} &= x \\ \int y\ dy &= \int\ dx \\ \frac{1}{2}y^2 &= x + c \\ y^2 &= 2x + d \end{aligned}$$

h) $x\sqrt{y^2-1} - y\sqrt{x^2-1}\frac{dy}{dx} = 0$

$$\begin{aligned} x\sqrt{y^2-1} - y\sqrt{x^2-1}\frac{dy}{dx} &= 0 \\ x\sqrt{y^2-1} &= y\sqrt{x^2-1}\frac{dy}{dx} \\ \int \frac{x}{\sqrt{x^2-1}}\ dx &= \int \frac{y}{\sqrt{y^2-1}}\ dy \\ \frac{1}{2}\sqrt{x^2-1} + c &= \frac{1}{2}\sqrt{y^2-1} \\ \sqrt{x^2-1} + d &= \sqrt{y^2-1} \end{aligned}$$

2) Use integrating factors to solve the following:

a) $\frac{dy}{dx} - 2xy = 2x$

The integrating factor is:

$$e^{\int -2x\ dx} = e^{-x^2}$$

Therefore:

$$\begin{aligned} \frac{dy}{dx} - 2xy &= 2x \\ e^{x^2}\frac{dy}{dx} - 2e^{x^2}xy &= 2e^{x^2}x \\ \frac{d}{dx}\left(e^{x^2}y\right) &= 2e^{x^2}x \\ e^{x^2}y &= \int 2e^{x^2}x\ dx \\ &= e^{x^2} + c \\ y &= 1 + ce^{-x^2} \end{aligned}$$

b) $\dfrac{dy}{dx} = y - x$

$$\begin{aligned} \frac{dy}{dx} &= y - x \\ \frac{dy}{dx} - y &= x \end{aligned}$$

The integrating factor is:

$$e^{\int -1 \; dx} = e^{-x}$$

Therefore:

$$\begin{aligned} \frac{dy}{dx} - y &= x \\ e^{-x}\frac{dy}{dx} - e^{-x}y &= e^{-x}x \\ \frac{d}{dx}\left(e^{-x}y\right) &= e^{-x}x \\ e^{-x}y &= \int e^{-x}x \; dx \\ e^{-x}y &= -xe^{-x} + \int e^{-x} \; dx \\ e^{-x}y &= -xe^{-x} - e^{-x} + c \\ y &= -x - 1 + ce^{x} \end{aligned}$$

c) $\dfrac{dy}{dx} + y\tan x = 1$

The integrating factor is:

$$\begin{aligned} e^{\int \tan x \; dx} &= e^{-\ln|\cos x|} \\ &= \frac{1}{\cos x} \\ &= \sec x \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{dy}{dx} + y\tan x &= 1 \\ \sec x\frac{dy}{dx} + y\sec x\tan x &= \sec x \\ \frac{d}{dx}(y\sec x) &= \sec x \\ y\sec x &= \int \sec x \; dx \\ y\sec x &= \ln|\sec x + \tan x| + c \end{aligned}$$

d) $\dfrac{dy}{dx} + ay = e^x$ (a is a constant)

The integrating factor is:

$$e^{\int a\ dx} = e^{ax}$$

Therefore:

$$\begin{aligned} \frac{dy}{dx} + ay &= e^x \\ e^{ax}\frac{dy}{dx} + e^{ax}ay &= e^{ax}e^x \\ \frac{d}{dx}\left(e^{ax}y\right) &= e^{(a+1)x} \\ e^{ax}y &= \int e^{(a+1)x}\ dx \\ e^{ax}y &= \frac{e^{(a+1)x}}{a+1} + c \\ y &= \frac{e^x}{a+1} + ce^{-ax} \end{aligned}$$

e) $x\dfrac{dy}{dx} - ay = x + 1$

$$\begin{aligned} x\frac{dy}{dx} - ay &= x + 1 \\ \frac{dy}{dx} - \frac{a}{x}y &= \frac{x+1}{x} \end{aligned}$$

The integrating factor is:

$$\begin{aligned} e^{\int -\frac{a}{x}\ dx} &= e^{-a\ln x} \\ &= x^{-a} \end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{dy}{dx} - \frac{a}{x}y &= \frac{x+1}{x} \\
x^{-a}\frac{dy}{dx} - x^{-a}\frac{a}{x}y &= x^{-a}\frac{x+1}{x} \\
\frac{d}{dx}\left(x^{-a}y\right) &= x^{-a-1}(x+1) \\
\frac{d}{dx}\left(x^{-a}y\right) &= x^{-a} + x^{-a-1} \\
x^{-a}y &= \int x^{-a} + x^{-a-1}\ dx \\
x^{-a}y &= \frac{x^{-a+1}}{-a+1} + \frac{x^{-a}}{-a} + c \\
y &= \frac{x}{1-a} - \frac{1}{a} + cx^{a}
\end{aligned}$$

f) $1 - ye^x = e^x \dfrac{dy}{dx}$

$$\begin{aligned}
1 - ye^x &= e^x \frac{dy}{dx} \\
e^x \frac{dy}{dx} &= 1 - ye^x \\
\frac{dy}{dx} &= e^{-x} - y \\
\frac{dy}{dx} + y &= e^{-x}
\end{aligned}$$

The integrating factor is:

$$e^{\int 1\ dx} = e^x$$

Therefore:

$$\begin{aligned}
\frac{dy}{dx} + y &= e^{-x} \\
e^x \frac{dy}{dx} + e^x y &= 1 \\
\frac{d}{dx}\left(e^x y\right) &= 1 \\
e^x y &= x + c \\
y &= xe^{-x} + ce^{x}
\end{aligned}$$

g) $t^2 \dfrac{dx}{dt} + xt + 1 = 0$

$$
\begin{aligned}
t^2\frac{dx}{dt} + xt + 1 &= 0 \\
\frac{dx}{dt} + \frac{1}{t}x &= -\frac{1}{t^2}
\end{aligned}
$$

The integrating factor is:

$$
\begin{aligned}
e^{\int \frac{1}{t}\ dt} &= e^{\ln t} \\
&= t
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\frac{dx}{dt} + \frac{1}{t}x &= -\frac{1}{t^2} \\
t\frac{dx}{dt} + x &= -\frac{1}{t} \\
\frac{d}{dt}(tx) &= -\frac{1}{t} \\
tx &= \int -\frac{1}{t}\ dt \\
tx &= -\ln t + c \\
x &= -\frac{\ln t}{t} + \frac{c}{t}
\end{aligned}
$$

h) $\cos^2 x\frac{dy}{dx} + y = \tan x$

$$
\begin{aligned}
\cos^2 x\frac{dy}{dx} + y &= \tan x \\
\frac{dy}{dx} + \sec^2 xy &= \sec^2 x\tan x
\end{aligned}
$$

The integrating factor is:

$$
e^{\int \sec^2 x\ dx} = e^{\tan x}
$$

Therefore:

$$
\begin{aligned}
\frac{dy}{dx} + \sec^2 xy &= \sec^2 x\tan x \\
e^{\tan x}\frac{dy}{dx} + e^{\tan x}\sec^2 xy &= e^{\tan x}\sec^2 x\tan x \\
\frac{d}{dx}\left(e^{\tan x}y\right) &= e^{\tan x}\sec^2 x\tan x \\
e^{\tan x}y &= \int e^{\tan x}\sec^2 x\tan x\ dx
\end{aligned}
$$

Let $u = \tan x$.

$$\frac{du}{dx} = \sec^2 x$$
$$du = \sec^2 x \ dx$$

Therefore:

$$e^{\tan x} y = \int e^{\tan x} \sec^2 x \tan x \ dx$$
$$e^{\tan x} y = \int e^u u \ du$$
$$e^{\tan x} y = ue^u - \int e^u \ du$$
$$e^{\tan x} y = ue^u - e^u + c$$
$$e^{\tan x} y = \tan x e^{\tan x} - e^{\tan x} + c$$
$$y = \tan x - 1 + ce^{-\tan x}$$

3) Newton's Law of Cooling states that

$$\frac{dT}{dt} + kT = k\theta,$$

where T is the temperature (in °C) of a body at time t (seconds), θ is the ambient temperature (a constant, measured in °C) and k is a constant.

In an experiment, a body starts at 10°C. After 10 seconds, it has reached 20°C. After 20 seconds, it has reached 25°C.

3a Show that $e^{-10k} = \dfrac{\theta - 20}{\theta - 10}$ and $e^{-20k} = \dfrac{\theta - 25}{\theta - 10}$.

First solve the ODE:

$$\frac{dT}{dt} + kT = k\theta$$
$$\frac{dT}{dt} = k\theta - kT$$
$$\frac{dT}{dt} = k(\theta - T)$$
$$\int \frac{1}{\theta - T} \ dT = \int k \ dt$$
$$-\ln|\theta - T| = kt + c$$
$$\ln|\theta - T| = -kt - c$$
$$\theta - T = Ae^{-kt}$$
$$T = \theta - Ae^{-kt}$$

We are given that $T(0) = 10$. This means that:

$$10 = \theta - AA \qquad = \theta - 10$$

We are given that $T(10) = 20$. This means that:

$$\begin{aligned} 20 &= \theta - Ae^{-10k} \\ Ae^{-10k} &= \theta - 20 \\ e^{-10k} &= \frac{\theta - 20}{A} \\ &= \frac{\theta - 20}{\theta - 10} \end{aligned}$$

We are given that $T(20) = 25$. This means that:

$$\begin{aligned} 25 &= \theta - Ae^{-20k} \\ Ae^{-20k} &= \theta - 25 \\ e^{-20k} &= \frac{\theta - 25}{A} \\ &= \frac{\theta - 25}{\theta - 10} \end{aligned}$$

3b Show that $\left(\frac{\theta - 20}{\theta - 10}\right)^2 = \frac{\theta - 25}{\theta - 10}$.

$$\begin{aligned} \left(e^{-10k}\right)^2 &= e^{-20k} \\ \left(\frac{\theta - 20}{\theta - 10}\right)^2 &= \frac{\theta - 25}{\theta - 10} \end{aligned}$$

3c Find the ambient temperature.

$$
\begin{aligned}
\left(\frac{\theta - 20}{\theta - 10}\right)^2 &= \frac{\theta - 25}{\theta - 10} \\
(\theta - 20)^2 &= (\theta - 25)(\theta - 10) \\
\theta^2 - 40\theta + 400 &= \theta^2 - 35\theta + 250 \\
-40\theta + 400 &= -35\theta + 250 \\
400 - 250 &= (40 - 35)\theta \\
150 &= 5\theta \\
30 &= \theta \\
\theta &= 30
\end{aligned}
$$