MATH6103 Differential & Integral Calculus

MATH6500 Elementary Mathematics for Engineers

Problem Sheet 4 Solutions

1) Find $\frac{dy}{dx}$ when:

a) $y = \sin^{-1} x$

$$\begin{aligned} y &= \sin^{-1} x \\ x &= \sin y \\ \frac{dx}{dy} &= \cos y \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ \frac{dy}{dx} &= \frac{1}{\cos \sin^{-1} x} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

b) $y = \ln x + \sec x$

$$\begin{aligned} y &= \ln x + \sec x \\ &= \ln x + (\cos x)^{-1} \\ \frac{dy}{dx} &= \frac{1}{x} + -1 \cdot (\cos x)^{-2} \cdot -\sin x \\ &= \frac{1}{x} + \frac{\sin x}{\cos^2 x} \end{aligned}$$

c) $y = \sin^{-1}(e^x)$

Using the chain rule and (a):

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x \\ &= \frac{e^x}{\sqrt{1-e^{2x}}} \cdot e^x \end{aligned}$$

d) $y = \frac{1}{\ln x}$

$$
\begin{aligned}
y &= (\ln x)^{-1} \\
\frac{dy}{dx} &= -1 \cdot (\ln x)^{-2} \cdot \frac{1}{x} \\
&= -\frac{1}{x(\ln x)^2}
\end{aligned}
$$

e) $y = e^x + 4^x$

$$
\begin{aligned}
y &= e^x + e^{x \ln 4} \\
\frac{dy}{dx} &= e^x + e^{x \ln 4} \cdot \ln 4 \\
&= e^x + 4^x \ln 4
\end{aligned}
$$

f) $y = \ln(x^4)$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x^4} \cdot 4x^3 \\
&= \frac{4x^3}{x^4} \\
&= \frac{4}{x}
\end{aligned}
$$

g) $y = 4 \ln x$

$$
\begin{aligned}
\frac{dy}{dx} &= 4 \cdot \frac{1}{x} \\
&= \frac{4}{x}
\end{aligned}
$$

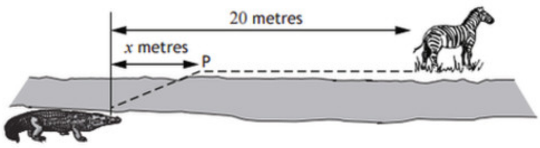
h) $y = x^x$

$$
\begin{aligned}
y &= e^{x \ln x} \\
\frac{dy}{dx} &= e^{x \ln x} \cdot \frac{d}{dx}(x \ln x) \\
\frac{dy}{dx} &= e^{x \ln x} \cdot \left(\ln x + \frac{x}{x}\right) \\
\frac{dy}{dx} &= x^x \cdot (\ln x + 1) \\
\frac{dy}{dx} &= x^x (\ln x + 1)
\end{aligned}
$$

2) A crocodile is stalking prey located 20 metres further upstream on the opposite bank of a river.

Crocodiles travel at different speeds on land and in water.

The time taken for the crocodile to reach its prey can be minimised if it swims to a particular point, P, x metres upstream on the other side of the river as shown in the diagram.



The time taken, T, measured in tenths of a second is given by

$$T(x) = 5\sqrt{36 + x^2} + 4(20 - x)$$

Calculate:

a) the time taken if the crocodile does not travel on land.

If the crocodile does not travel on land, then $x = 20$.

$$\begin{aligned} T(20) &= 5\sqrt{36 + 20^2} + 4(20 - 20) \\ &= 5\sqrt{36 + 400} + 0 \\ &= 5\sqrt{436} \\ &= 20\sqrt{109} \end{aligned}$$

b) the time taken if the crocodile swims the shortest distance possible.

If the crocodile swims the shortest distance possible, then $x = 0$.

$$\begin{aligned} T(0) &= 5\sqrt{36 + 0^2} + 4(20 - 0) \\ &= 5\sqrt{36} + 80 \\ &= 30 + 80 \\ &= 110 \end{aligned}$$

c) the value of x such that the time T is minimised and hence the minimum possible time.

$$
\begin{aligned}
T(x) &= 5\sqrt{36+x^2} + 4(20-x) \\
&= 5(36+x^2)^{\frac{1}{2}} + 80 - 4x \\
T'(x) &= 5 \cdot \frac{1}{2}(36+x^2)^{\frac{-1}{2}} \cdot 2x - 4 \\
&= \frac{5x}{\sqrt{36+x^2}} - 4
\end{aligned}
$$

The minimum will occur when $T'(x) = 0$:

$$
\begin{aligned}
T'(x) &= 0 \\
\frac{5x}{\sqrt{36+x^2}} - 4 &= 0 \\
\frac{5x}{\sqrt{36+x^2}} &= 4 \\
5x &= 4\sqrt{36+x^2} \\
25x^2 &= 16(36+x^2) \\
25x^2 &= 576 + 16x^2 \\
9x^2 &= 576 \\
x^2 &= \frac{576}{9} \\
x &= \frac{24}{3} \\
x &= 8
\end{aligned}
$$

Therefore the minimum possible time is $T(8)$:

$$
\begin{aligned}
T(2\sqrt{8}) &= 5\sqrt{36+8^2} + 4(20-8) \\
&= 5\sqrt{36+64} + 4 \cdot 12 \\
&= 5\sqrt{100} + 48 \\
&= 5 \cdot 10 + 48 \\
&= 50 + 48 \\
&= 98
\end{aligned}
$$